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A Long-Ring Embedding Scheme in the Faulty Star Graph

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ABSTRACT

The star graph interconnection network has been recognized as an attractive alternative to the hypercube network. In this paper, we investigate in the faulty star graph the ring embedding problem. It has been shown that a ring containing at least $n! - 4f$ nodes can be embedded in an n-dimensional star graph or n-star graph with $f \le n - 3$ faulty nodes, where $n!$ is processor number of the n-star graph. In this paper, a long-ring embedding scheme is proposed which can be used to embed a ring with at least $n! - 2f$ nodes in an n-star graph to achieve tolerance of up to the same number of faulty nodes. Our results outperform those presented by Y.C. Tseng and colleagues in 1997.

Key Words: fault tolerance, graph embedding, interconnection network, ring, star graph

I. Introduction

A new interconnection network that has recently attracted substantial attention is the star graph (Akers and Krishnameurthy, 1989; Akers *et al*., 1987). The star graph, being a member of the class of Cayley graphs, has been shown to possess appealing features, including a small number of nodes, a small diameter, partitionabilty, symmetry, and a high degree of fault tolerance. Accordingly, much research has been done on the star graph's topological properties (Day and Tripathi, 1994; Qiu *et al*., 1994), embedding capability (Jwo *et al*., 1991; Nigam *et al*., 1990; Tseng *et al*., 1997, 1999), communication capability (Nigam *et al*., 1990; Mendia and Sarkar, 1992; Akl *et al*., 1993; Mišić *et al*., 1994; Qiu *et al*., 1994; Fragopoulou and akl, 1996; Sheu *et al*., 1995), and fault-tolerance (Bagherzadeh *et al*., 1993; Latifi, 1993; Jovanović and Mišić, 1994; Tseng *et al*., 1997; Chen and Sheu, 2000).

One critical issue in evaluating a network is the *graph embedding problem*. Given a *guest graph G* and a *host graph H*, the problem is to find a mapping from each node of G to one of H, and a mapping from each edge of G to one path in H. This problem has long been used to model the problem of arranging a parallel algorithm in a parallel architecture. The graph embedding problem has been heavily studied for various host graphs. Rings are common guest graphs with many applications. With a star graph as the host graph, it has been shown that any ring of even length is embedable (Jwo *et al*., 1991). Results for embedding multi-dimensional meshes into a star graph can be found in Jwo *et al*. (1991) and Qiu *et al*. (1994).

Fault tolerance is an important issue, especially when the size of the star graph system increases, since a large system is required in order to continue to perform operations after failure of one or more processors/links. In this paper, we consider the problem of embedding a ring into a faulty star graph. This paper will focus on the node-fault model. In this model, faulty nodes are assumed to neither perform calculations nor route data. Further, our model can be extended to an edge-fault model in the following way. An edge fault is assumed to exist when one of the nodes incident upon it is assumed to be faulty. If some components fail in a star graph, it is desirable for the faulty components to be isolated from the rest of the network so that embedding will still be possible. The similar problem of fault-tolerant ring embedding in hypercubes has also been studied by Chan and Lee (1991).

The fault-tolerant ring embedded scheme was initially proposed by Tseng *et al*. (1997). They presented a top-down embedding approach to constructing a ring containing at least $n! - 4f$ nodes in S_n if there are $f \le n - 3$ faulty nodes. In contrast to the top-down embedding approach (Tseng *et al*., 1997), our embedding scheme uses a bottom-up approach. This embedding approach is also significant since the main concept behind it is to build a concatenation tree to concatenate small sub-rings into large rings. To do this, a tree-based concatenation scheme is introduced. The significant feature of this approach is the embedding of a ring whose length is at least $n! - 2f$ into

S_n, where $f \le n - 3$. The result is an improvement over previous methods proposed by Tseng *et al*. (1997), and it leads to a near optimal result because the star graph is a bipartite graph (Akers *et al*., 1987). Notably, it is an optimal result when $f = 1$.

The rest of this paper is organized as follows. The preliminary and basic ideas are introduced in Section II. A base-ring embedding scheme is presented in Section III. A generalized technique for ring embedding into a faulty star graph is addressed in Section IV. Finally, conclusions are drawn in Section V.

II. Preliminary

This section will introduce the host graph model and some basic ideas necessary for our embedding scheme.

1. Star Graph

An n-dimensional star graph, also referred to as an n-star or S_n, is an undirected graph consisting of $n!$ nodes (or vertices) and $(n-1)n!/2$ edges. To each node is uniquely assigned a label $x_1x_2\cdots x_n$, which is the concatenation of a permutation of n distinct symbols $\{x_1, x_2, \cdots, x_n\}$. Without loss of generality, let these n symbols be $\{1, 2, ..., n\}$. Given any node label $x_1\cdots x_i\cdots x_n$, let the permutation function g_i, $2 \le i \le n$, be such that $g_i(x_1\cdots x_i\cdots x_n) = x_i\cdots x_1\cdots x_n$ (i.e., exchange x_1 and x_i, and keep the other symbols unchanged). In S_n, for any node x, there is an edge joining x and node $g_i(x)$, and the direction of this edge is along dimension i. Each node in S_n is connected to $n-1$ adjacent nodes by $n-1$ edges. Each S_n contains n disjoint S_{n-1}'s. An S_4 is illustrated in Fig. 1.

Let $\Gamma = \{1, 2, \ldots, n, *\}$, where * denotes a *don't care* symbol. Every substar of S_n can be uniquely labeled by a string of symbols in Γ, such that the only repeated symbol is *. Formally, a k-dimensional substar, S_k or k-substar, is denoted as a string $G = x_1x_2\cdots x_n$, and the number of * symbols in string G is k, where $x_1 = *$ and $x_i \in \Gamma$, $2 \le i \le n$. The substar represented by G is a subgraph of S_n, containing all the vertices obtained from G by replacing each * with digits $\{1, 2, \ldots, n\}$. These vertices are connected by the original links in S_n. For instance, **3*1 is a 3-dimensional substar containing the set of nodes {54321, 45321, 52341, 25341, 42351, 24351}. Throughout this paper, a k-substar is said to be faulty if there exists at least one faulty node in the k-substar, where $1 \le k \le n$. Otherwise, the k-substar is said to be fault-free.

We will now describe two useful notations, j-split and D-split operations, for the partition scheme. Let $G = x_1x_2\cdots x_j\cdots x_n$ be a k-substar with $x_j = *$. The j-split operation is applied on G, $2 \le j \le n$, which is used to partition G along the j-dimension into k copies of $(k-1)$-substars, each obtained from G by replacing x_j with a legal non-* symbol. Let $D = \{d_1, d_2, \ldots, d_m\}$, $m \le k$, be a set of dimensions such that $x_{d_i} = *$, $i = 1 \cdots m$. Then the D-split operation is set to perform m times of j-split operations on G, where $j \in D$, as follows. We begin by applying a d_1-split operation on G, to the result of which we then apply a d_2-split operation, to the result of which we then apply a d_3-split operation, etc, until there is $k(k-1)\cdots(k-m+1)$ number of $(k-m)$-substars. Notice that the value of j can not be 1 since if $j = 1$, then the partitioning result does not retain a complete set of substars.

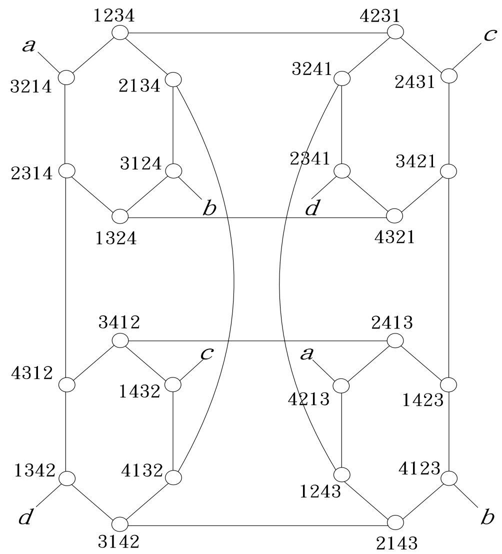

Fig. 1. A star graph S_4.

Given any two k-substars, $G = x_1x_2\cdots x_i\cdots x_n$ and $H = y_1y_2\cdots y_i\cdots y_n$ are said to be adjacent if and only if the labels of G and H differ in exactly one dimension, where $1 < i \le n$. If G and H are adjacent, the *difference* between G and H, denoted as $dif(G,H)$, is the symbol of G at the position where G and H differ. For example, substar G = ***13 is adjacent to H = ***12 and H' = ***23, but not adjacent to H'' = ***32. The difference between H and G, or $dif(H,G) = 2$, and the difference between H' and G, or $dif(H',G) = 2$. Given a sequence of adjacent k-substars $[G_0, G_1, \cdots, G_{t-1}]$, a (k,t)-**ring** must be defined first before we can construct our ring. A sequence of k-substars $[G_0, G_1, \cdots, G_{t-1}]$ is denoted as a (k,t)-**ring** if substar G_i is adjacent to its neighboring $G_{(i-1)\ mod\ t}$ and $G_{(i+1)\ mod\ t}$, and $dif(G_{(i-1)\ mod\ t}, G_i) \ne dif(G_{(i+1)\ mod\ t}, G_i)$, for any $i = 0\ldots t-1$. For example, [****2, ****4, ****5, ****1, ****3] is a (4,5)-**ring**, but [***32, ***12, ***13, ***23, ***21, ***31] is not a (3,6)-**ring** since dif(***12, ***13) = dif(***23, ***13) = 2.

2. Concatenation Operation

This subsection will define a basic operation, referred to as the concatenation operation. We will initially give a lemma for the concatenation operation.

Lemma 1. *Given a* (k,t)-**ring** = $[G_0, G_1, \cdots, G_{t-1}]$, *where* $t = 3$ *or* 4, *there are at most* $(k-1)!$ *pairs of edge disjoint cycles with length* $2t$ *such that each cycle is constructed by*

$$P_0 \leftrightarrow P_1 \leftrightarrow P_2 \leftrightarrow \cdots P_{2i-1} \leftrightarrow P_{2i} \leftrightarrow P_{2i+1} \leftrightarrow P_{2i+2}$$

$$\leftrightarrow P_{2t-1} \leftrightarrow P_0,$$

where neighboring nodes P_{2i} *and* $P_{2i+1} \in G_i$ *and* $0 \le i \le t-1$.

Proof. Given a (k,t)-**ring**, let three k-substars, $G_{i-1} = x_1x_2\cdots x_n$, $G_i = y_1y_2\cdots y_n$, and $G_{i+1} = z_1z_2\cdots z_n$, be neighboring substars. Let $\alpha = dif(G_{i-1},G_i)$, $\alpha' = dif(G_i,G_{i-1})$, $\beta = dif(G_{i+1},G_i)$, and $\beta' = dif(G_i,G_{i+1})$, where $\alpha \ne \beta$. There are $(k-1)!$ pairs of nodes of $P_{2i} = \alpha y_2\cdots y_n$ of G_i directly connected to corresponding nodes of $P_{2i-1} = \alpha' x_2\cdots x_n$ of G_{i-1}, and there are $(k-1)!$ pairs of nodes of $P_{2i+1} = \beta y_2\cdots y_n$ of G_i directly connected to corresponding nodes of $P_{2i+2} = \beta' z_2\cdots z_n$ of G_{i+1}. Intuitively, there are $(k-1)!$ pairs of nodes of $P_{2i} = \alpha y_2\cdots\beta\cdots y_n$ connected to nodes of $P_{2i+1} = \beta y_2\cdots\alpha\cdots y_n$ when g_i is applied, where $y_i = \alpha$. If $t = 3$, then $(k-1)!$ cycles are formed by $P_0 \overset{g_i}{\leftrightarrow} g_i(P_0) \overset{g_n}{\leftrightarrow} g_n(g_i(P_0)) \overset{g_i}{\leftrightarrow} g_i(g_n(g_i(P_0))) \overset{g_n}{\leftrightarrow} g_n(g_i(g_n(g_i(P_0)))) \overset{g_i}{\leftrightarrow} g_i(g_n(g_i(g_n(g_i(P_0))))) \overset{g_n}{\leftrightarrow} P_0$, where $2 \le i \le n-1$. Note that g_n is one of the feasible selections. For simplicity, in the following discussion, we will only use g_n to determine the construction of a ring. This must be correct because each cycle is isomorphic to a cycle as shown in Fig. 2(c) and is denoted as (P_0,i,n,i,n,i,n,P_0). When $t = 4$, $(k-1)!$ cycles exist within the path $(P_0,i,n,j,n,i,n,j,n,P_0)$, where $i \ne j$ and $2 \le i,j \le n-1$. Intuitively, the path $(P_0,i,n,j,n,i,n,j,n,P_0)$ is isomorphic to one cycle, as illustrated in Fig. 2(d). □

For example, consider a (4,3)-**ring** = [****2, ****4, ****1] such that *dif*(****2, ****4) = 2, *dif*(****4, ****2) = 4, *dif*(****1, ****4) = 1, and *dif*(****4, ****1) = 4. There are six pairs of disjoint cycles, 1***2 $\overset{g_i}{\leftrightarrow}$ 4***2 $\overset{g_5}{\leftrightarrow}$ 2***4 $\overset{g_i}{\leftrightarrow}$ 1***4 $\overset{g_5}{\leftrightarrow}$ 4***1 $\overset{g_i}{\leftrightarrow}$ 2***1 $\overset{g_5}{\leftrightarrow}$ 1***2, where $2 \le i \le 4$. For a further example, a (4,4)-**ring** = [****2, ****4, ****1, ****3] is illustrated in Fig. 2(b).

Now, we can precisely define the concatenation operation. Using the concatenation operation, we can embed a larger ring in a faulty S_n, using a bottom-up approach. From Lemma 1, there are $(k-1)!$ pairs of disjoint cycles in $(k,3)$-**ring** and $(k,4)$-**ring**. The definition of (k,t)-**ring** is a general definition. For a specified exact subring used in the our ring construction, it is customary to use the term $R_{k,t}$ to represent the exact sub-ring. Note that $R_{k,t}$ is any one of $(k-1)!$ ring pairs in (k,t)-**ring**. The detailed defini-

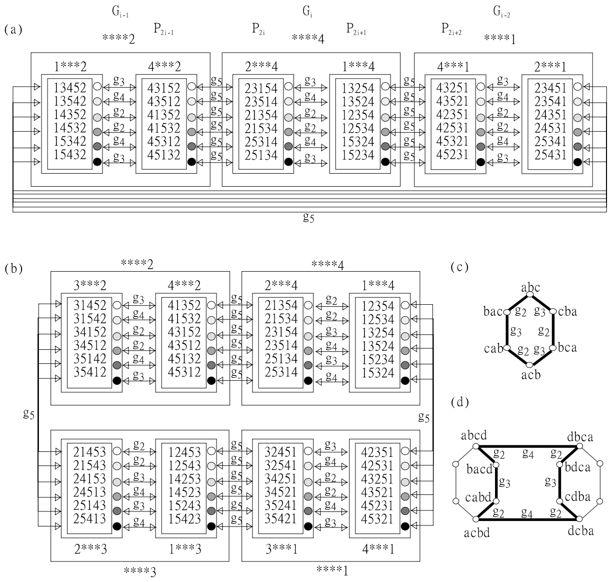

Fig. 2. Six pairs of disjoint cycles in (a) a (4,3)-ring, (b) a (4,4)-ring, and a distinct cycle in (c) a 3-star and in (d) a 4-star.

tion of $R_{k,t}$ is as follows. Let each cycle of $(k,3)$-**ring** and $(k,4)$-**ring** be denoted as $R_{k,3}$ and $R_{k,4}$, respectively. Note that $R_{k,3}$ or $R_{k,4}$ are used to construct a larger ring by concatenating three or four disjoint existed sub-rings. The larger ring also retains the total number of nodes in all of these sub-rings. The operation is described as follows. Assume that there are three disjoint rings R_1, R_2, and R_3; we can combine R_1, R_2, and R_3 into one larger ring. As shown in Fig. 3(a), assume that there is $R_{k,3} = \{P_0,P_1,P_2,P_3,P_4,P_5,P_0\}$ between three rings such that $P_0 \leftrightarrow P_1$, $P_2 \leftrightarrow P_3$, and $P_4 \leftrightarrow P_5$ are edges of R_1, R_2, and R_3, respectively. Intuitively, there is one other path from P_0 to P_1, denoted as $\widehat{P_0P_1}$, with length $|R_1| - 1$, since R_1 is a ring. Similarly, the paths $\widehat{P_2P_3}$ and $\widehat{P_4P_5}$ can be constructed in R_2 and R_3. Then a larger ring whose path length is $|R_1| + |R_2| + |R_3|$ can be established by $\widehat{P_0P_1} \leftrightarrow \widehat{P_2P_3} \leftrightarrow \widehat{P_4P_5} \leftrightarrow P_0$. Such a concatenating operation uses one $R_{k,3}$ to concatenate three disjoint rings. Similarly, we can use one $R_{k,4}$ to concatenate four disjoint rings, as shown in Fig. 3(b).

Now, we will give an important lemma for our embedding scheme.

Lemma 2. *Assume that there are three and four adjacent substars* S_k*'s, in each of which is embedded a ring with* $k!$ *nodes,* $k \ge 4$. *At least* $R_{k,3}$ *and* $R_{k,4}$ *may exist, which can concatenate three and four existing disjoint rings into a larger ring, respectively.*

Proof. Intuitively, as shown in Fig. 3(a), if $k = 4$, then there exist at least six distinct $R_{k,3}$ which can be used to

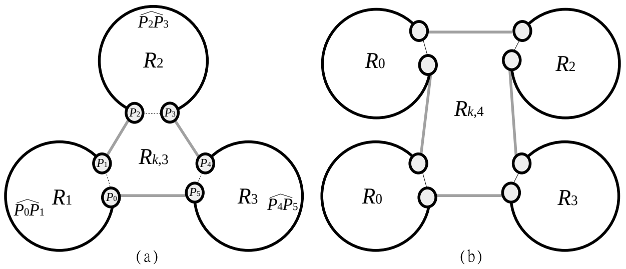

Fig. 3. (a) $R_{k,3}$ and (b) $R_{k,4}$ concatenate three and four disjoint rings into larger rings, respectively.

perform the concatenation operation. Furthermore, if $k > 4$, then $(k-1)!$ $R_{k,4}$ exists. As shown in Fig. 3(b), if $k = 4$, since R_0 can be directly adjacent to R_1 and R_2, or to R_1 and R_3, to R_2 and R_3, each condition has 6 possible $R_{3,4}$ which can be used to perform the concatenation operation, so in total there exist 18 possible $R_{3,4}$. Clearly, if $k > 4$, then there exist $3(k-1)!$ possible $R_{k,4}$ which can be used to perform the concatenation operation. □

3. Concatenation Tree

Before introducing the embedded scheme, a *concatenation tree* T_σ will be introduced based on lemma 2. Concatenation tree T_σ is used to concatenate m disjoint rings into a ring without node loss, where the tree height $\sigma = \lceil \log_4(m) \rceil$. To simplify our description of how to construct a T_σ, we will give an assumption for the concatenation operation as follows. Given any three or four disjoint rings, there exists a feasible $R_{k,3}$ or $R_{k,4}$ which can be used to form a large ring without any node loss. In what follows, we will present an embedding scheme which satisfies our assumption. Using this embedding scheme, we can correctly construct our embedding ring. Initially, we define a function $Ns(m)$ as follows. First, if $m < 9$, then $Ns(m)$ produces a number sequence as shown in Fig. 4(a). Secondly, if $m \geq 9$, then $Ns(m)$ produces a number sequence which satisfies the following conditions: (1) all the elements are arranged in descending order, (2) the total number of elements is equal to m, and (3) each element is equal to 3 or 4. For instance, 44333 is a number sequence. The function $Ns(m)$ is defined as follows.

$Ns(m)$: The number of number sequences is equal to $r = \lceil m/4 \rceil$. Let the first $m - 3r$ elements be equal to 4, and let all of the remaining elements be equal to 3.

For example, if $m = 17$, then $r = \lceil 17/4 \rceil = 5$, so the number sequence is 44333. If there are m disjoint rings, and if the length of each ring is l_i, where $1 \leq i \leq m$, then T_σ is recursively constructed according to the following steps.

C1: If $m = 3$ or 4, then we use one $R_{k,3}$ or $R_{k,4}$ to concatenate three or four distinct sub-rings into one and then stop the construction operation.

C2: If $5 \leq m \leq 8$, then we let $r = \lfloor (m-1)/3 \rfloor$ and use r $R_{k,3}$ to produce r rings. If the number of remaining rings is $m - 2r = 3$ or 4, then we perform **C1**. For instance, if $m = 7$, then we use two $R_{k,3}$ to construct two disjoint rings, so there are three distinct rings.

C3: We produce a number sequence $Ns(m)$ if $m \geq 9$. If each element of the number sequence is equal to 3 or 4, then we apply $r = \lceil m/4 \rceil$ concatenation operations by using $R_{k,3}$ or $R_{k,4}$ to form r disjoint rings. We repeatedly perform steps **C1** to **C3** after setting m to be r. For instance, $Ns(17)$ produces 44333, so we obtain five disjoint rings.

After the above steps are completed, a T_σ is established through a bottom-up method. Our ring embedding scheme is implemented based on the construction of T_σ as follows. All m disjoint rings are viewed as leaf nodes of T_σ while every three and four leaf nodes, determined by $Ns(m)$, can form a ring. This operation correspondingly forms a branch/parent node (an upper level of the tree) from three or four leaf nodes. The concatenation operations begins at the last level of T_σ. Then m_1 disjoint rings are formed, where $m_1 < m$. The construction operations are continued and produce m_2 disjoint rings, where $m_2 = Ns(m_1)$. This constructs the upper level of T_σ. Thus, m_1 disjoint rings produce m_2 disjoint rings, where $m_2 < m_1$. Concatenation operations are repeatedly executed until the number of rings is equal to one. Therefore, a ring with length $= \Sigma_{i=1,m} l_i$ can be constructed. For example, given 17 disjoint rings, a 3-level T_σ tree is shown in Fig. 4(b). Notice that we assume that the root of T_σ is in level 0.

III. Embed Base-Ring in Faulty S_n When n Is Smaller Than 6

In the following sections, we will study the following problem: given an S_n with f faulty nodes, find a ring that is as large as possible without passing any faulty

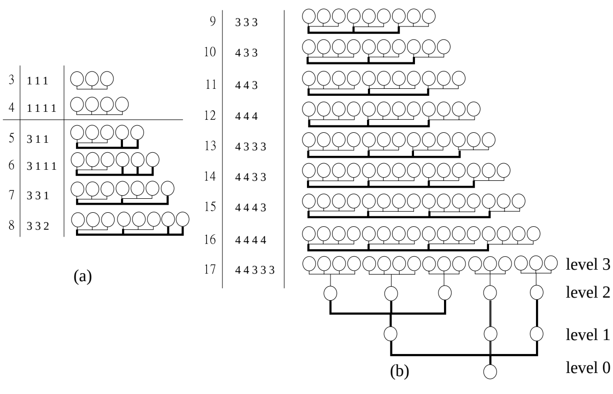

Fig. 4. (a) T_σ trees, where $3 \leq m \leq 8$, and (b) T_σ trees, where $m \geq 9$ and $\sigma = \lceil \log_4(m) \rceil$.

node. Our main result shows that for any $f \le n - 3$, a ring at least $n! - 2f$ in length can be found. This result is an improvement over the results of Tseng *et al*. (1997). We will divide the discussion into the following cases, depending on the value of n. Note that the path length in a ring is exactly equal to the number of nodes used in the ring.

We will present a lemma for the partitioning scheme used in the faulty S_n with $f \le n - 3$. In Lemma 3 we will show that, given an S_n with $f \le n - 3$ faulty nodes, there always exists a D-split operation on S_n, as defined in Section II.1, such that each S_4 contains at most one faulty node. A ring which is embedded in a faulty S_k, $4 \le k \le n$, with at most $f = k - 3$ faults is denoted as an X_k-ring. The length of this X_k-ring is at least $k! - 2f$. Initially, our base embedding is implemented to embed a ring, namely X_4-ring, in S_4 with at most one fault. Our algorithm is a recursive algorithm which repeatedly constructs an X_{k+1}-ring on S_{k+1} if all the X_k-rings of $k + 1$ S_k's are built in advance.

Lemma 3. *In an S_n, $n \ge 4$, with $f \le n - 3$ faulty nodes, there always exists a D-split operation, $|D| = n - 4$, on S_n which results in 4-substars, each containing at most one faulty node* (Tseng *et al*., 1997).

For example, consider an S_7 with the faulty set F = {1234567, 1342567, 4312567, 4321657}. We will examine it from position 7 to position 2. A 7-split will not work since all the faulty nodes will fall into one 6-substar. Therefore, we apply a 6-split, which partitions F into the subsets F_1 = {1234567, 1342567, 4312567} and F_2 = {4321657}. Next, we need to partition F_1. However, a 5-split will not work, so we apply a 4-split, which partitions F_1 into the subsets F_{11} = {1234567} and F_{12} = {1342567, 4312567}. Finally, a 3-split can partition F_{12} into two subsets. Therefore, a D-split with D = (6, 4, 3) is the desired split.

An S_n can be decomposed into $n(n - 1)\cdots(k + 2)$ S_{k+1}'s by applying a D-split operation in S_n. Each S_{k+1} can be further decomposed into $(k + 1)$ S_k's by applying a j-split in S_{k+1}. After applying a split operation on a S_{k+1}, there are $k + 1$ copies of S_k. Assume that in each S_k can be embedded a X_k-ring. In the following, we will describe how to recursively construct an X_{k+1}-ring from $k + 1$ existing X_k-rings, where $n > k \ge 4$. By repeatedly applying the above process, an X_n-ring in S_n with faults can be established. Our embedding scheme is a bottom-up process; that is, a larger embedding ring in S_k is constructed by means of smaller embedded sub-rings in all S_{k-1}'s.

1. Construct a Ring in S_4 with One Fault

We will first explain how to construct an X_4-ring on S_4 with one fault node. Note that, if S_4 has no faults, a ring can be constructed (Jwo *et al*., 1991; Nigam *et al*., 1990; Tseng *et al*., 1997). Assume that a faulty node of S_4 is $X = uvwx$. Apply a 4-split operation on S_4 to obtain 3-substars $***u$ and $***v$, $***w$ and $***x$. Denote $D = g_4(X) \in ***u$ as a *dangling node*, which is a nonfaulty node but is not used to form this X_4-ring. All of the remaining nodes of S_4, except for the faulty node X and dangling node D, are used to form the X_4-ring as shown in Fig. 5. The X_4-ring is established starting from nodes $U = g_2(X)$ and $V = g_3(X)$, where nodes X, U and $V \in ***x$. A path $\in ***x$ is constructed by $(U, 3,2,3,2, V)$. Dangling node D and nodes $U' = g_2(D)$ and $V' = g_3(D)$ are located in $***u$, and a path $\in ***u$ with length 5 is connected by means of $(U', 3,2,3,2, V')$. The neighboring node $U'' = g_4(U)$ of node U can form a ring $\in ***v$ with length 6 by means of $(U'', 3,2,3,2,3, g_2(U''))$. Similarly, the neighboring node $V'' = g_4(V)$ of node V can form a ring $\in ***w$ with length 6 by means of $(V'', 2,3,2,3,2, g_3(V''))$. Since node $g_2(U'')$ connects with U' and $g_3(V'')$ connects with V', an X_4-ring with length 22 is built. Let $g_2(U'') = g_2(g_4(U)) = g_2(g_4(g_2(X)))$, and let $U' = g_2(D) = g_2(g_4(X))$. Intuitively, $g_4(g_2(g_4(g_2(X))))$ is equal to $g_2(g_4(X))$; then nodes $g_2(U'')$ and U' are neighboring along dimension 4. Similarly, nodes $g_3(V'')$ and V' are neighboring along dimension 4. Therefore, our X_4-ring with length 22 is constructed. Two properties of the process used to construct the X_4-ring are stated as follows.

A1: All nodes of 3-substars $***v$ and $***w$ are used to form an X_4-ring. One node is not used to form an X_4-ring on 3-substars $***x$ (faulty node *uvwx*) and $***u$ (dangling node *xvwu*), respectively.

A2: There are six edges in total on 3-substars $***v$ (and $***w$). There are two pairs of links between $u**v$ with $x**v$ of $***v$ (and between $u**w$ with $x**w$ of $***w$), but one pair of links between $u**v$ with $x**v$ is not used in constructing an X_4-ring. (This guarantees that we can correctly construct a larger ring.)

For instance, given an S_4 with F = {4231}, node

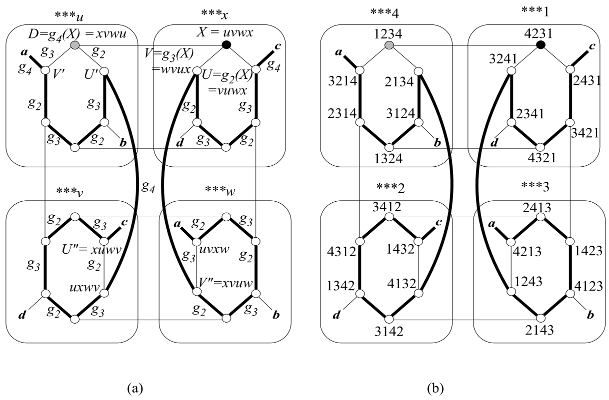

Fig. 5. An X_4-ring on S_4 with faulty node 4231.

1234 = g_4(4231) is a *dangling node*. Nodes 4231 and 1234 are unused, and ***1, ****2, ***3, and ****4 are obtained by applying 4-split on S_4. Note that all the nodes of ***2 and ***3 are used. The X_4-ring is constructed starting from nodes 2431 and 3241. A path ∈ ***1 from node 2431 to 3241 is constructed as 2431 ↔ 3421 ↔ 4321 ↔ 2341 ↔ 3241. Intuitively, dangling node 1234, node 2134 = g_2(1234) and node 3214 = g_3(1234) are all located in ***4, and a path ∈ ***4 with length 5 is constructed by means of (2134, 3,2,3,2, 3214). Neighboring node 1432 of 2431 ∈ ***2 forms a ring with length 6 by means of (1432, 3,2,3,2,3, 4132). Moreover, neighboring node 1243 of 3241 ∈ ***3 forms a ring with length 6 by means of (1243, 2,3,2,3,2, 4213). Node 4132 connects with 2134, and 4213 connects with 3214, so an X_4-ring is, therefore, established. Therefore, we have the following lemma.

Lemma 4. *A base embedding an X_4-ring with length* 22 *exists on S_4 with one fault.*[1]

2. Construct a Base-Ring in S_5 with Two Faults

Based on Lemma 3, after applying a feasible *D*-split operation on S_n, we can determine that each S_k contains at most k – 3 faulty nodes, and that each S_k contains at least three fault-free S_{k-1}, where $4 \le k \le n$. For example, each S_5 has at most two faulty nodes and at least three fault-free S_4's. In the following, we will describe how to correctly establish a base-ring, an X_5-ring, from five X_4-rings by using two $R_{4,3}$.

We will initially describe the conditions of an X_5-ring with two faults. The embedded ring will be constructed from five possible base embedding X_4-rings. If an S_5 has two faults, it can be decomposed into five S_4's, which have at most two S_4's with one fault each. Based on Lemma 4, a base embedded X_4-ring is constructed on each S_4. Assume that an X_5-ring is constructed by means of a substar sequence = [G_0, G_1, G_2, G_3, G_4], where G_0 = ****b_0, G_1 = ****b_1, G_2 = ****b_2, G_3 = ****b_3, and G_4 = ****b_4. The substar sequence must satisfy the following conditions.

B1: [G_0, G_1, G_2] and [G_2, G_3, G_4] are (4,3)-**ring**s.

B2: Substars G_0, G_2, and G_4 are nonfaulty substars. Each substar, G_1 or G_3, contains one fault.

Based on the properties of **Al** and **A2**, we can select feasible substars G_0, G_2, and G_4 to guarantee that an X_5-ring can be constructed. First, assume that there exist two faulty nodes $f = uvwxb_1 \in G_1$ and $f' = u'v'w'x'b_3 \in G_3$. By **Al**, all the nodes of 3-substars ***vb_1 and ***wb_1 of G_1 (***$v'b_3$ and ***$w'b_3$ of G_3) are used to form its X_4-ring. By **A2**, in G_1, links between u**vb_1 and x**vb_1 of (u**wb_1 and x**wb_1) of G_1 are unused. Similarly, one of two links between u'**$v'b_3$ and x'**$v'b_3$ (u'**$w'b_3$ and x'**$w'b_3$) of G_3 is unused. Therefore, adjacent substars of G_1 and G_3 are determined by satisfying the **B3** or **B4** conditions.

B3: Substars G_0 and G_2 are not simultaneously equal to ****u and ****x.

B4: Substars G_2 and G_4 are not simultaneously equal to ****u' and ****x'.

After determining the location relationship between substars G_0, G_2, and G_4, we then set two (4,3)-**ring**s to be [G_0, G_1, G_2] and [G_2, G_3, G_4]. Clearly, an X_5-ring can be constructed by using these two $R_{4,3}$. The length of the X_5-ring is equal to 5! – 4. For example, an X_5-ring in a faulty S_5 with two faults is illustrated in Fig. 6.

Lemma 5. *An X_5-ring with length* 5! – 2*f exists in S_5 with f faults, where $f \le 2$.*

Proof. According to the above descriptions, an X_5-ring with length 5! – 4 exists on S_5 with two faults. Similarly, if an S_5 contains one fault, then a ring with length 5! – 2 is constructed from one X_4-ring and four non-faulty rings in fault-free S_4's by means of two $R_{4,3}$. □

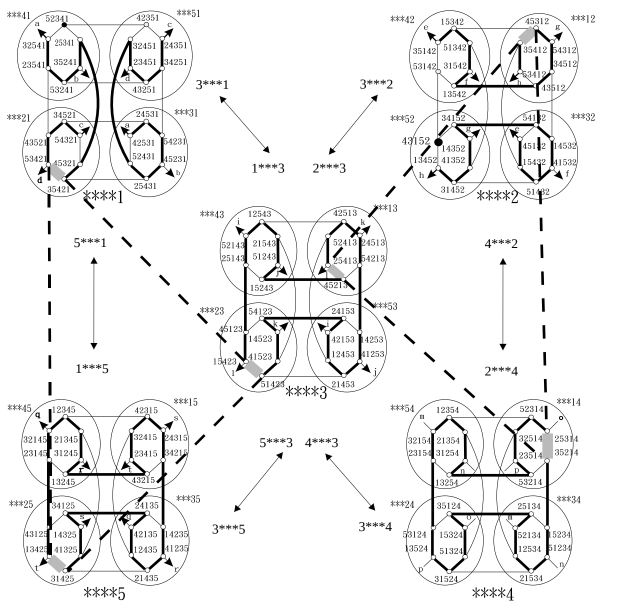

Fig. 6. Constructing an X_5-ring on faulty S_5 with two faults using two $R_{4,3}$'s.

[1] The result is optimal for the case f = 1 since a star graph is bipartite.

IV. Embed a Ring on a Faulty S_n When $n \geq 6$

In Section III.2, one feasible X_5-ring in a 5-substar was constructed. All possible constructed X_5-rings are used to form a large ring. In this section, we will present a generalized approach to embedding a long-ring with $n! - 2f$ nodes in an S_n based on using the concatenation tree T_σ, where $\sigma = \lceil \log_4(n-3) \rceil$ and $n \geq 6$.

Given any S_n with $n - 3$ faults, there exist at least three fault-free S_{n-1}'s and at most $n - 3$ faulty S_{n-1}'s. As described above, an S_n can be decomposed into n copies of S_{n-1} in which each pair of S_{n-1} is adjacent. Each faulty S_{n-1} is assumed to contain an X_{n-1}-ring in the worse case. Note that an X_n-ring is established if we can correctly connect $n - 3$ X_{n-1}-rings with three fault-free S_{n-1}'s. For the induction hypothesis, assume the existence of an X_{n-1}-ring in an S_{n-1}. We will show how to correctly embed an X_n-ring concatenated by $n - 3$ X_{n-1}-rings and three fault-free (unembedded) S_{n-1}'s in S_n. This embedding process is divided into two steps:

(1) Connect $n - 3$ X_{n-1}-rings by means of three fault-free S_{n-1}'s such that $n - 3$ disjoint rings are obtained (**D1** and **D2** operations),

(2) Concatenate the final $n - 3$ disjoint rings into one large ring without node loss by using a concatenation tree (**D3** and **D4** operations).

Step 1 is initially stated as follows. Given $n - 3$ faulty S_{n-3}'s, then $n - 3$ X_{n-1}-rings are assumed to be constructed on each faulty S_{n-3}. The goal is to produce $n - 3$ distinct rings, denoted as $\tilde{X}_{n-1}$-rings, and to connect these $n - 3$ $\tilde{X}_{n-1}$-rings with three fault-free S_{n-1}'s. Two operations are performed to construct $n - 3$ X_{n-1}-rings as follows.

D1: Let $\hat{X}_{n-1}$-rings be X_{n-1}-rings with the following modification. Each $\hat{X}_{n-1}$-ring is constructed by performing concatenation operations on each X_{n-1}-ring with three S_{n-2}'s by using an $R_{n-2,4}$, where each S_{n-2} belongs to a distinct fault-free S_{n-1}. This yields $n - 3$ disjoint $\hat{X}_{n-1}$-rings. (See the example shown in Fig. 7(a).) This ring must be connected since these three distinct S_{n-2}'s are used for the first time.

D2: Let $\tilde{X}_{n-1}$-rings be $\hat{X}_{n-1}$-rings with the following modification. For each original fault-free S_{n-1}, there are two remaining unembedded $(n-2)$-substars. Choose one $\hat{X}_{n-1}$-ring from **D1**, and connect it to these two $(n-2)$-substars by using a feasible $R_{n-2,3}$. (See the example shown in Fig. 7(b).) Similarly, this ring must be connected since these distinct S_{n-2}'s are used for the first time.

For example, as shown in Fig. 7, three X_5-rings are constructed in *****1, *****2, *****3. Substars *****4, *****5, and *****6 are fault-free. Consider an X_5-ring existing in *****1 with one edge selected from ****21 such that an $R_{4,4}$ is established from {****21, ****24, ****25, ****26}. Notice that this $R_{4,4}$ can be from either

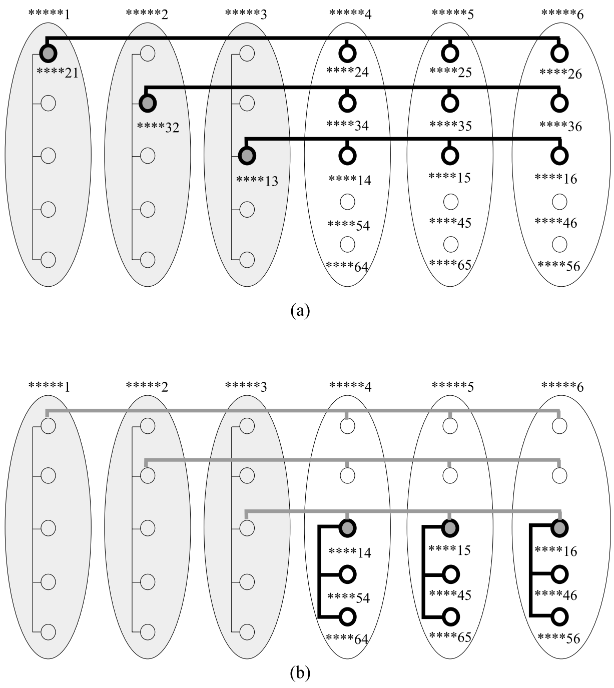

Fig. 7. Constructing three (a) $\hat{X}_5$-rings and (b) $\tilde{X}_5$-rings in a faulty S_6.

{****21, ****24, ****26, ****25} or {****21, ****25, ****24, ****26}. Similarly, three $\hat{X}_5$-rings are established. For 4-stars ****14, ****15, and ****16, we respectively construct three $R_{4,3}$ {****14, ****54, ****64}, {****15, ****45, ****65}, and {****16, ****46, ****56} to concatenate six unembedded substars, ****54, ****64, ****15, ****45, ****16, and ****46, in order to construct $\tilde{X}_5$-rings.

The following lemma indicates the correctness of the construction of $n - 3$ disjoint $\tilde{X}_{n-1}$-rings.

Lemma 6. *There are* $n - 3$ *disjoint* $\tilde{X}_{n-1}$*-rings in an* S_n *with at most* $n - 3$ *faulty nodes*.

Proof. Without loss of generality, consider that there are $n - 3$ faulty S_{n-1}'s, and that into each one has already been embedded an $\hat{X}_{n-1}$-ring. On each $\hat{X}_{n-1}$-ring is performed a concatenation operation with three $(n-2)$-substar by means of an $R_{n-2,4}$, where each $(n-2)$-substar belongs to one of three fault-free $(n-1)$-substars. Suppose that three fault-free S_{n-1}'s are $*^{n-1}a$, $*^{n-1}b$, and $*^{n-1}c$; then all the other S_{n-1}'s can be denoted as $*^{n-1}x$, where $x \in \{1, \ldots, n\} - \{a, b, c\}$. A $(n-1)$-split operation is applied on each of $*^{n-1}a$, $*^{n-1}b$, and $*^{n-1}c$, so each one has $n - 1$ S_{n-2}'s. At least one $R_{n-2,4}$ can be found from $[*^{n-2}wx, *^{n-2}wy_1, *^{n-2}wy_2, *^{n-2}wy_3]$, where $y_1 \neq y_2 \neq y_3$ and y_1, y_2, and $y_3 \in \{a, b, c\}$. One edge is selected from a substar $*^{n-2}wx$, where an X_{n-1}-ring is in a faulty substar $*^{n-1}x$. All the edges of substars $*^{n-2}wa$, $*^{n-2}wb$, and $*^{n-2}wc$ are fault-free and are not used now. Let two neighboring substars of $*^{n-2}wx$ be $*^{n-2}wy_1$ and $*^{n-2}wy_3$, where $(y_1,y_3) = (a,b)$ or (a,c) or (b,c). Owing to the fact that there are $3*(n-3)!$ possible selections if there are no faults, there must exist one edge between one pair of $z_1*^{n-3}wx$ and $z_2*^{n-3}wx$, where $(z_1,z_2) = (a,b)$ or (a,c) or (b,c). Therefore, $n-3$ disjoint $\hat{X}_{n-1}$-rings can be constructed. In addition, for each original fault-free $(n-1)$-substar, there are two remaining unembedded $(n-2)$-substars. Therefore, $n-3$ disjoint $\tilde{X}_{n-1}$-rings are made using an $R_{n-2,4}$. This completes the proof. □

We will now describe step 2. Given $n-3$ $\tilde{X}_{n-1}$-rings from Lemma 6, let $\sigma = \lceil \log_4(n-3) \rceil$, and let a concatenation tree T_σ be constructed and used to hierarchically concatenate $n-3$ $\tilde{X}_{n-1}$-rings into a larger ring. This concatenation operation is divided into two steps as follows.

D3: Tree T_σ is used to concatenate $n-3$ $\tilde{X}_{n-1}$-rings as follows. During construction of a T_σ, we first perform a concatenation operation on level σ by using a sequence number determined by $Ns(n-3)$ so as to produce s disjoint rings. This operation is executed on one of three fault-free S_{n-1}.

D4: The concatenation operation is repeatedly performed on levels $\sigma-1$, $\sigma-2$, $\cdots$, and 1, and applied on two other original fault-free S_{n-1}'s in turn. Each operation is determined by an $Ns(s')$ function, where s' is the current number of rings. Note that s' represents the total number of rings in level i, where $i \in \sigma-1, \sigma-2, \cdots$, and 1. Each concatenation operation is performed by finding feasible $R_{n-2,3}$'s or $R_{n-2,4}$'s in order to concatenate three or four rings into one.

To prove the correctness of the embedding, we will give an important lemma as follows.

Lemma 7. *Given three or four adjacent substars S_k's into each of which is assumed to be embedded a ring that is k! nodes in length. There must at least exist two pairs of $R_{k,3}$'s or $R_{k,4}$'s to simultaneousely concatenate these three or four subrings (each one is existed in a S_k) into a large one, where $k \geq 4$.*

Proof. Recall Lemma 1; if $k = 4$, then there exist at least 6 possible $R_{k,3}$ is which can be used to perform the concatenation operation. In the worse case, three substar S_3's can destroy at most three possible $R_{4,3}$'s, so there at least will exist one possible $R_{4,3}$. This case can be verified, for $k > 4$, because that the growth in the total number of $R_{k,3}$'s (by a factorial factor of k) is higher than the growth of the number of destroyed $R_{k,3}$'s (by a linear factor of k). Similarly, this reason also applies to the case of $R_{k,4}$. This completes the proof. □

That is, we have the following result.

Corollary 1. *Given any fault-free substar S_k, it is possible to perform $R_{k,3}$ or $R_{k,4}$ twice to concatenate subrings.*

For example, Fig. 6 shows an X_5-ring that is constructed by means of a ring located in substar ****3, which uses two $R_{4,3}$'s to concatenate four other subrings. Based on Corollary 1, we can show the correctness of the following result.

Lemma 8. *There exists a σ-level concatenation tree T_σ which can be used to concatenate $n-3$ disjoint $\tilde{X}_{n-1}$-rings into a larger ring, where $\sigma = \lceil \log_4(n-3) \rceil$ and $n \geq 6$.*

Proof. In the **D3** operation, a concatenation operation is executed on level σ by means of a sequence number determined by $Ns(n-3)$ to produce s disjoint rings. Note that these s disjoint rings must be carefully constructed as follows.

D3': Let $\hat{X}'_{n-1}$-rings be $\hat{X}_{n-1}$-rings if the concatenation operation can be applied on level σ on the rightmost S_{n-1}. Otherwise, let $\hat{X}'_{n-1}$-rings can be reconstructed by means of a new $R_{n-2,4}$ (the same condition as in **D1**).

Note that all of the S_{n-2}'s in the rightmost S_{n-1} satisfy Corollary 1. Therefore, the concatenation operation can produce s disjoint rings in the rightmost S_{n-1} as shown in Fig. 8.

In the **D4** operation, the concatenation operation then is applied on levels $\sigma-1$, $\sigma-2$, $\cdots$, and 1. These concatenation operations are, in turn, performed on the other two S_{n-1}'s as follows. Notably, all the S_{n-2}'s of these two S_{n-1}'s will also satisfy Corollary 1. We can explain this as follows. Initially, in the **D1** and **D2** operations, all the S_{n-2}'s of the two S_{n-1}'s have already used one $R_{n-2,3}$ or $R_{n-2,4}$. In the following, we will show how to perform the concatenation operations on levels $\sigma-1$, $\sigma-2$, $\cdots$, and 1 on these S_{n-2}'s such that these S_{n-2}'s will use $R_{n-2,3}$ or $R_{n-2,4}$ once only.

For ease of presentation, an example will be used to illustrate the above operation for the case $n = 18$. After executing a **D3** or **D3**' operation, 15 disjoint $\tilde{X}_{n-1}$-rings will be correctly combined into 5 larger subrings in the rightmost S_{17} as shown in Fig. 8. A $R_{16,3}$ is selected in the second S_{17} in order to concatenate 3 subrings. Note that this $R_{16,3}$ is selected from three groups, which are located in the second S_{17}. Each group has three or four embedded rings, and each one has already used the concatenation operation once (as in the **D1** operation). After this, three subrings are formed. Therefore, an $R_{16,3}$ is chosen in the third S_{17} in order to concatenate the final 3 subrings into one ring. Note that this $R_{16,3}$ is selected from three distinct groups in the third S_{17}. Similarly, each one has three or four embedded rings, and each one already has used the concatenation operation once (as in the **D1** and **D2** operations). Therefore, a final ring can be constructed by using

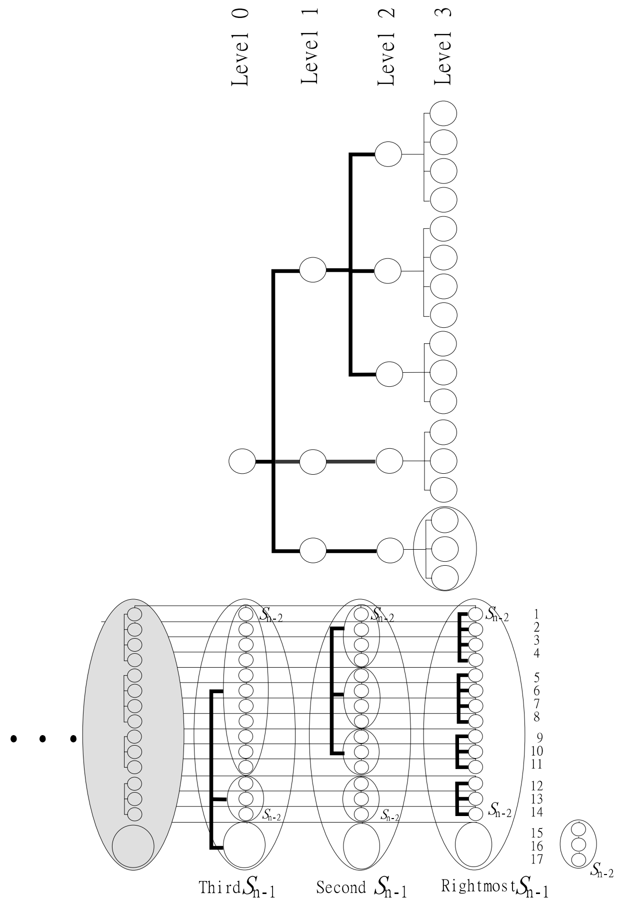

Fig. 8. Constructing three (a) $\hat{X}_5$-rings and (b) $\tilde{X}_5$-rings in a faulty S_6.

a constructed T_3.

In general, $R_{n-2,3}$'s or $R_{n-2,4}$'s are selected from three or four groups of substars in the second and third S_{n-1}'s in turn. Notice that in each group there at least exists one embedded ring which only uses the concatenation operation once. This is because the number of embedded rings which use one concatenation operation will increase during each concatenation step in levels $\sigma - 1$, $\sigma - 2$, $\cdots$, and 1. Clearly, it is guaranteed that all of the connected subrings satisfy Corollary 1. Therefore, we can concatenate $n - 3$ disjoint $\tilde{X}_{n-1}$-rings into a larger ring by constructing T_σ, where $\sigma = \lceil \log_4(n-3) \rceil$ and $n \geq 6$. □

Theorem 1. *An S_n with f faults can embed an X_n-ring whose length is at least $n! - 2f$, where $f \leq n - 3$.*

V. Conclusions

In this paper, we have proposed an improved method for finding a long ring in a faulty star graph S_n. The star graph can establish a ring with $n! - 4f$ nodes in a star graph with f faulty nodes, where $f \leq n - 3$, as proposed by Tseng *et al*. (1997). Our improved method constructs a long ring with $n! - 2f$ nodes. The result is a great improvement over the method of Tseng *et al*. (1997). Work is currently underway to develop a method to embed a larger ring in a faulty star graph when the number of faulty nodes is more than $n - 3$.

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一個在受損的星狀網路上嵌入較長的環之策略

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摘 要

近來星狀連結網路被視爲取代超維立方體網路的一種極受重視的網路架構。在此篇論文中，我們探討了一個在受損的星狀網路上嵌入較長的環之策略。目前文獻上最好的結果是在含有f個錯誤處理器的n維度的星狀網路可以嵌入一個長度爲$n!-4f$的環，其中$n!$代表n維度的星狀網路的處理器個數。此篇論文提出一個改進之較長的環之嵌入策略，使得在含f個錯誤處理器的n維度的星狀網路可以嵌入一個長度爲$n!-2f$的環，而經由本篇論文的證明確實改善之前的研究成果。