Toward Optimal Complete Exchange on Wormhole-Routed Tori

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Abstract—In this paper, we propose new routing schemes to perform all-to-all personalized communication (or known as complete exchange) in wormhole-routed, one-port tori. On tori of equal size along each dimension, our algorithms use both asymptotically optimal startup and transmission time. The results are characterized by several interesting features: 1) the use of gather-scatter tree to achieve optimality in startup time, 2) enforcement of shortest paths in routing messages to achieve optimality in transmission time, 3) application of network-partitioning techniques to reduce the constant associated with the transmission time, and 4) the dimension-by-dimension and gather-scatter-tree approach to make possible applying the results to nonsquare, any-size tori. In the literature, some algorithms are optimal in only one of startup and transmission costs, while some, although asymptotically optimal in both costs, will incur much larger constants associated with the costs. Numerical analysis and experiment both show that significant improvement can be obtained by our scheme on total communication latency over existing results.

Index Terms—All-to-all personalized communication, broadcast, complete exchange, gossiping, multicomputer network, torus, wormhole routing.

1 INTRODUCTION

A dvances in technology have made possible multi-computers of large scale. In a multicomputer network, fast and efficient interprocessor communication is crucial to unleashing the aggregated computing power. The most basic communication pattern is one-to-one (unicast). Recent research has put much attention on the collective communication, which incurs denser and heavier traffic on the network. Examples include one-to-all (broadcast), one-to-many (multicast), and all-to-all communications and a large amount of work can be found in [2], [4], [5], [7], [11], [15], [16], [17], [18], [20], [22], [30]. Messages to be sent can be further classified as nonpersonalized (wherein all receivers will receive a same message from a same source) and personalized (wherein each receiver will receive a different message from a same source). Some of these communication patterns have also been implemented in PVM [8] and MPI [19] as communication libraries.

In this paper, we study the all-to-all personalized communication, or known as complete exchange or gossiping, wherein each node needs to send a distinct message to each of the rest of the nodes. This represents the densest communication pattern among what is identified above. Applications of complete exchange include matrix algorithms, fast Fourier transformation (FFT), graph algorithms, and data redistribution in HPF [13]. It can also be used to evaluate the quality of an interconnection network. Previous work for complete exchange can be found in [3], [9], [23], [27], [28], [29] for meshes and [6], [10], [25], [26], [31], [32] for tori.

Here, the torus network is considered, which architecture has been adopted by commercial machines such as Cray T3D/T3E. The switching model under consideration is wormhole routing, which has been widely used in existing machines such as Caltech MOSAIC, Cray T3D/T3E, IBM SP2, Intel Touchstone Delta, Intel Paragon, MIT J-machine, and nCUBE3.

Works related to the problem considered in this paper include [1], [6], [10], [14], [25], [26], [31], [32]. The results in [1], [6], [14] are based on a torus/mesh using packet switching (or store-and-forward). Such schemes are inappropriate for wormhole-routed networks as the distance-insensitive property is hardly exploited. Communication in a wormhole-routed network typically incurs two kinds of costs: startup time and transmission time.

Both schemes in [10], [31] use the optimal transmission time to achieve complete exchange in a torus. However, the startup cost is pretty high—$O(n^3)$ in a 2D $n \times n$ torus and $O(n^4)$ in a 3D $n \times n \times n$ torus. To relieve this problem, reference [32] proposes a diagonal-propagation scheme which uses asymptotically optimal transmission time, but incurs a much lower $O(n)$ startup time (for both 2D and 3D tori). This startup time is still relatively higher than the theoretical lower bound of $O(\lg n)$. The first scheme that is known to use both asymptotically optimal startup time and transmission time is proposed in [25], [26]. However, the constant associated with the transmission time is relatively high and the effect of this is significant as the amount of...
data sent in complete exchange is fairly large (see the comparison in Section 6).

We comment that the complete-exchange algorithms developed for meshes [3], [9], [23], [27], [28], [29] may be directly applied on tori. However, such algorithms may fail in using the additional bandwidth provided by tori (a torus has twice the bisection bandwidth that of a mesh of the same size) and, thus, are inherently slower than good torus algorithms, as has been observed by [32].

In this paper, we also present a complete exchange scheme which uses asymptotically optimal startup and transmission time. For a brief overview, refer to Table 3 and Table 4. Our 2D and 3D schemes both incur transmission time of \( \Theta(n^2) \) times the lower bound, as opposed to that of \( \Theta(2^n) \) and 10 times, respectively, the lower bound in [25], [26]. According to our numerical evaluation, significant gain can be achieved by our schemes (refer to Fig. 9 and Fig. 11 for a quick overview).

In addition to performance gain, our schemes also possess some features which are worth of pointing out. First, inspired by [25], we also use a “gather-then-scatter” (or called bottom-up in [25]) technique to achieve asymptotically optimal startup time. Second, we try to send messages along shortest paths as much as possible. This turns out to be important to achieve optimality in transmission time. On the contrary, references [25], [26], [31] use nonminimal paths to deliver messages. Third, inspired by [34], [35], we adopt the network-partitioning technique to divide a torus into multiple logical subtori. This turns out to be helpful for our schemes to fully utilize the communication bandwidth and to conform to the one-port model, wherein a node can only send, and simultaneously receive, one worm at a time. Last, we take a dimension-by-dimension and gather-scatter-tree approach, which makes easy extending our schemes to any-dimen- sional, nonsquare, non-power-of-2 tori (which seems to be difficult, if not impossible, for the approaches adopted by [25], [26], [32]).

The rest of this paper is organized as follows. As a basic construct, Section 2 develops a complete exchange scheme on a 1D ring. Based on this construct, we present our complete exchange schemes for 2D and 3D tori in Section 3 and Section 4, respectively. The extensions to nonsquare, non-power-of-2 tori are discussed in Section 5. Some numerical analysis and evaluation are shown in Section 6 to demonstrate the strength of our result. In Section 7, issues of synchronization in our schemes are discussed. Conclusions are drawn in Section 8.

## 2 Basic Construct: Complete Exchange on a Ring

In this section, we consider the complete exchange problem on a ring of length \( n = 2^n \). Nodes on the ring are denoted as \( v_i \), \( i = 0, \ldots, (n-1) \). Between \( v_i \) and \( v_{(i+1) \mod n} \), there is a positive link from \( v_i \) to \( v_{(i+1) \mod n} \) and a negative link along the reverse direction. The positive distance from \( v_i \) to \( v_j \), denoted as \( \text{dist}^+(v_i, v_j) \), equals \((j-i) \mod n\) and the negative distance from \( v_i \) to \( v_j \), denoted as \( \text{dist}^-(v_i, v_j) \), is \( n - \text{dist}^+(v_i, v_j) \). Below, we omit saying “mod” whenever the context is clear. On the ring, a transmission from \( v_i \) to \( v_j \) along the positive direction will be denoted as \( v_i \leftarrow v_j \), while that along the negative direction will be denoted as \( v_j \rightarrow v_i \).

In the problem of complete exchange, each node \( v_i \) has a message block (or simply block) denoted as \( b_i \), which is aimed at node \( v_i \). We use \( b_i^{s,j} \) to denote the set of blocks \( \{b_i, b_i^{s+1}, \ldots, b_i^{s+j}\} \) and \( b_i^{s,j} \) the set of blocks \( \{b_i, b_i^{s+1}, \ldots, b_i^{s+j}\} \). Symbols \( \leftarrow \) and \( \rightarrow \) are used in the incremental and decremental senses, respectively. Likewise, we define \( b_i^{s,j} = \{b_i^{s,j}, b_i^{s+1,j}, \ldots, b_i^{s+j}\} \) and \( b_i^{s,j} = \{b_i^{s,j}, b_i^{s+1,j}, \ldots, b_i^{s+j}\} \).

### 2.1 The Gather-Scatter Tree

Our scheme consists of a sequence of gathering phases followed by a sequence of scattering phases. In the beginning, all nodes will join the communication. After each gathering phase, the blocks are concentrated into a smaller number of nodes. On the contrary, blocks are distributed to more nodes after each scattering phase. At the end, it is guaranteed that every block arrives at its destination. The communication patterns of these phases are defined as follows:

**Definition 1.** Given any \( l, 0 \leq l \leq d - 2 \), define the communication phases \( GP_l^+ \) and \( SP_l^+ \) as follows:

\[
GP_l^+ = \{v_i \leftarrow v_{i+2^l} \mid i \mod 2^l = 0\}.
\]

In the definition, \( GP \) stands for “gathering phase,” \( SP \) for “scattering phase,” and \( + \) for “positive” direction. Note that although \( GP_l^+ \) and \( SP_l^+ \) have the same communication pattern, as yet to be shown, different blocks are delivered in them.

The concept of the so-called gather-scatter tree is best described by putting together a sequence of positive phases,

\[
GP_0^+ \rightarrow GP_1^+ \rightarrow \cdots \rightarrow GP_{l-2}^+ \rightarrow SP_{l-2}^+ \rightarrow SP_{l-3}^+ \rightarrow \cdots \rightarrow SP_0^+.
\]

An example is shown in Fig. 1 with \( d = 4 \). The gathering phases are time-spaced vertically from the bottom, while the scattering phases are time-spaced similarly from the top. We will call such a tree the positive gather-scatter tree (though, precisely speaking, it is a graph).

The height of the tree is \( d - 1 \). The tree is very helpful in determining how to route a block from one node to another, by taking some gathering phases followed by some scattering phases. For instance, three routes exist from \( v_2 \) to \( v_4 \): 1) \( v_2 \rightarrow v_4 \) in \( GP_2^+ \), 2) \( v_2 \rightarrow v_4 \) in \( SP_2^+ \), and 3) \( v_2 \rightarrow v_4 \) in \( GP_0^+ \) followed by \( v_3 \rightarrow v_4 \) in \( SP_0^+ \).

**Definition 2.** For each integer \( l, 0 \leq l \leq d - 2 \), define \( V_l = \{v_i \mid i \mod 2^l = 0\} \). For each integer \( l, 1 \leq l \leq d - 1 \), define \( V_l = V_{l-1} - V_l \) except that \( V_{d-1} = V_{d-2} \). For all values of \( l \) unspecified, \( V_l = \emptyset \) and \( V_0 = \emptyset \).

Intuitively, if \( v_i \) belongs to \( V_l \), it will join the communication in \( GP_l^+ \). However, \( v_i \) will not join the next gathering phase \( GP_{l+1}^+ \) if \( v_i \in V_{l+1} \). For example, \( v_3 \) belongs to \( V_2 \) implying that \( v_2 \) will communicate in \( GP_2^+ \), but not in \( GP_3^+ \) because \( v_2 \in V_2 \). On the contrary, \( v_3 \) is in \( V_1 \) and \( V_2 \), so...
it will communicate in both \(GP^+_i\) and \(GP^-_i\). Similar phenomena hold true for scattering phases.

The gather-scatter tree can also be used in determining the set of nodes from/to which a node can gather/scatter blocks. Specifically, if \(v_i \in V_t \cup \tilde{V}_t\), the set of nodes from which \(v_i\) can collect blocks in the gathering phases is ("C" means “coverage”)

\[
GC_i^+(v_i) = \{v_i, v_{i-1}, \ldots, v_{i-(2^k-1)}\}.
\]

Similarly, the set of nodes to which \(v_i\) can forward blocks in scattering phases is

\[
SC_i^+(v_i) = \{v_i, v_{i+1}, \ldots, v_{i+2^{k-1}}\}.
\]

This leads to the following lemma.

**Lemma 1.** For any \(v_i\) and \(v_t\) such that \(\text{dist}^+(v_i, v_t) \leq \frac{2^k}{2}\), there exists a path leading from \(v_t\) to \(v_i\) on the positive gather-scatter tree.

**Proof.** This can be validated since there always exists a \(G_m\) between \(v_t\) and \(v_i\) such that \(v_m \in V_t \cup \tilde{V}_t\) satisfying \(v_i \in GC_m^+(v_m)\) and \(v_t \in SC_m^+(v_m)\) for some \(h, 0 \leq h \leq (d-1)\).

**Definition 3.** Given any \(l, 0 \leq l \leq d-2\), define the communication phases \(GP^+_l\) and \(SP^-_l\) as follows:

\[
GP^+_l = SP^-_l = \{v_i \rightarrow v_{i-2^l} \mid i \mod 2^l = 0\}.
\]

Definition 3 is simply rewritten from Definition 1 by using links in the negative direction. It is easy to generalize to the concept of the negative gather-scatter tree (by reversing the directions of all transmissions in Fig. 1) and further prove a reachability property similar to Lemma 1. It should be understood that the extension to the negative tree is straightforward.

In Sections 2.2 and 2.3, we will develop a complete exchange scheme using the positive and negative gather-scatter trees. For each \(v_i\), it will deliver \(n/2\) blocks \(b_{i+2^k-1}^+\) on the positive tree, and \(n/2 - 1\) blocks \(b_{i-1}^-\) on the negative tree. However, as the negative tree is symmetric to the positive one, we will concentrate our discussion on the positive tree.

### 2.2 The Path Selection Strategy for Blocks

As shown earlier, there may exist multiple paths between a pair of source and destination nodes on the positive gather-scatter tree. How to choose from these paths to reduce the communication latency is a difficult problem. Mainly, we need a good heuristic to balance the communication load (number of transmitted blocks) on each link in a phase.

The following observation is used as a guideline in designing our scheme:

**Observation 1.** For two nodes \(v_i \in V_t\) and \(v_j \in \tilde{V}_t\), the traffic in \(v_i\) tends to be busier than that in \(v_j\) as \(v_i\) needs to join more communication phases than \(v_j\) does.

We next discuss a strategy for routing blocks in the gathering phases. Consider the gathering phase \(GP^+_i\), \(0 \leq i \leq (d-2)\). Suppose that, right before \(GP^+_i\), a block \(b_s^\prime\) has arrived at node \(v_i \in V_t\). We need to decide, in the communication \(v_i \rightarrow v_{i+2^l}\) in \(GP^+_i\), whether \(b_s^\prime\) should be sent to \(v_{i+2^l}\) or not. There are two cases:

**Case 1:** \(v_i \in \tilde{V}_{i+1}\). This implies that \(v_i\) will be prohibited from communicating in the subsequent gathering phases \(GP^+_i\) and scattering phases \(SP^+_l, l > i\), which in turn implies that the scattering coverage of \(v_i\) is at most as large as \(SC^+_l(v_i)\). Now, consider the location of \(v_i\) (see Fig. 2a for an illustration):

1. \(\text{dist}^+(v_i, v_i) < 2^l\), i.e., \(v_i \in SC^+_l(v_i)\). Apparently, \(b_s^\prime\) should not be sent to \(v_{i+2^l}\); otherwise, the block will go too far beyond its destination.
2. \(2^l \leq \text{dist}^+(v_i, v_i) < 2^{l+1}\). If so, \(v_i\) is in both \(SC^+_l(v_i)\) and \(SC^+_l(v_i+2^l)\). That is, \(v_i\) can be reached from both \(v_i\) and \(v_{i+2^l}\) using later scattering phases. Because \(v_i \in \tilde{V}_{i+1}\) and \(v_{i+2^l} \in \tilde{V}_{i+1}\), according to Observation 1, \(v_{i+2^l}\) tends to be busier than \(v_i\), and thus, \(b_s^\prime\) should not be sent in this phase.
3. \(\text{dist}^+(v_i, v_i) \geq 2^{l+1}\). This implies \(v_i \notin SC^+_l(v_i)\) and, thus, \(b_s^\prime\) must be sent in \(v_i \rightarrow v_{i+2^l}\) so as to reach \(v_i\).

**Case 2:** \(v_i \in V_{i+1}\). This implies that the receiving node \(v_{i+2^l}\) is in \(V_{i+1}\) and will not communicate in the subsequent gathering phases \(GP^+_i\) and scattering phases \(SP^+_l, l > i\), which in turn implies that the scattering coverage of \(v_{i+2^l}\) is at most as large as \(SC^+_l(v_{i+2^l})\). Now, consider the location of \(v_i\) (see Fig. 2b for an illustration):

1. \(\text{dist}^+(v_i, v_i) < 2^l\), i.e., \(v_i \in SC^+_l(v_i)\). Apparently, \(b_s^\prime\) should not be sent to \(v_{i+2^l}\).
2. \(2^l \leq \text{dist}^+(v_i, v_i) < 2^{l+1}\), i.e., \(v_i \in SC^+_l(v_{i+2^l})\). However, it is also possible that \(v_i \in SC^+_l(v_j)\) such...
that \( v_i \in V_t \) or \( \hat{V}_t, l' \geq l + 1 \). Observation 1 indicates that \( v_{i+2} \) will be less busy and, thus, \( b'_i \) should be sent in \( v_i \rightarrow v_{i+2} \).

3. \( \text{dist}^+(v_i, v_j) \geq 2^l + 2^{l+1} \). This implies \( v_i \notin SC_{i+1}^+(v_{i+2}) \) and, thus, \( b'_i \) should not be sent in this phase (it will join later phases for wider coverage).

Routing in the scattering phases is simpler. Consider the scattering phase \( SP_{i}^+ \). Suppose that, right before \( SP_{i}^+ \), a block \( b'_i \) has arrived at \( v_i \in V_{i+2} \cup \hat{V}_t \) (i.e., \( V_1 \)). If \( \text{dist}^+(v_i, v_j) < 2^l \), i.e., \( v_i \in SC_{i+1}^+(v_{i+2}) \), apparently \( b'_i \) should not be sent to \( v_{i+2} \) (or it will go too far beyond its destination). Otherwise, \( v_i \in SC_{i+1}^+(v_{i+2}) \) and, thus, \( b'_i \) should be sent in \( v_i \rightarrow v_{i+2} \).

**Example 1.** Fig. 3 shows the transmission patterns from the point of view of sources \( v_0, v_1, v_2, \) and \( v_3 \) to some destinations on the positive gather-scatter tree. For instance, consider the source \( v_1 \) in Fig. 3b. In \( GP_{0}^+ \), because \( v_1 \in V_t \), Case 1 should be applied to \( v_1 \rightarrow v_2 \). As destination \( v_2 \) satisfies \( 2^0 \leq \text{dist}^+(v_1, v_2) < 2^1 \), subcase 2 should be applied and, thus, \( b'_i \) should remain in \( v_1 \). As destination \( v_1, t = 3.9 \), satisfies \( \text{dist}^+(v_1, v_i) \geq 2^1 \), subcase 3 should be applied and, thus, \( b'_i \) (\( b'_{3.9} \)) should be sent to \( v_2 \). In \( GP_{0}^+ \), consider \( b'_{3.9} \) that has been moved to \( v_2 \). Because destination \( v_2 \in \hat{V}_t \), again Case 1 should be applied to \( v_2 \rightarrow v_3 \). As destination \( v_3 \) satisfies \( \text{dist}^+(v_2, v_3) < 2 \), subcase 1 should be applied and, thus, \( b'_i \) should remain in \( v_2 \). As destination \( v_2, t = 4.5 \), satisfies \( 2^1 \leq \text{dist}^+(v_2, v_j) < 2^2 \), subcase 2 should be applied and, thus, \( b'_{3.9} \) should remain in \( v_3 \). As destination \( v_3, t = 6.9 \), satisfies \( \text{dist}^+(v_2, v_j) \geq 2^2 \), subcase 3 should be applied and, thus, \( b'_{3.9} \) should be sent to \( v_3 \).

Now, consider the source \( v_3 \), as shown in Fig. 3d. Similar to the decisions for source \( v_1 \), in \( GP_{0}^+ \), \( b'_{3} \) should be moved to \( v_4 \) (Case 1). In \( GP_{1}^+ \), Case 2 should be applied to \( v_4 \rightarrow v_5 \). As \( v_5 \) satisfies \( \text{dist}^+(v_4, v_5) < 2 \), subcase 1 should be applied and, thus, \( b'_i \) should remain in \( v_4 \). As \( v_4, t = 6.9 \), satisfies \( 2^1 \leq \text{dist}^+(v_4, v_j) < 2^2 \), subcase 2 should be applied and, thus, \( b'_{3.9} \) should be sent to \( v_5 \). As \( v_5, t = 10.11 \), satisfies \( \text{dist}^+(v_4, v_j) \geq 2^1 + 2^2 \), subcase 3 should be applied and, thus, \( v_5 \) should remain in \( v_5 \).

**2.3 The Routing Algorithm**

We now reorganize the algorithm in a formal way. Routing on the positive tree consists of 2d − 2 phases:

\[
GP_{0}^+ \rightarrow GP_{1}^+ \rightarrow \ldots \rightarrow GP_{d-2}^+ \rightarrow SP_{d-2}^+ \\
SP_{d-3}^+ \rightarrow \ldots \rightarrow SP_{0}^+.
\]

Initially, each \( v_i \) has a pool of blocks \( B_i = b'_{i-2^l} \). Note that \( B_i \) will change by time. At the end of the algorithm, the \( B_i \) in each \( v_i \) contains \( b'_{i-2^l} \). Every \( v_i \) executes the following phases synchronously.

**Phase GP_{i}^+:** /\( l = 0, 1, \ldots, (d - 2) \).

if \( v_i \in V_t \) then

if \( v_i \in V_{i+2} \) then //Case 2.

\[ M = \{ b'_i | b'_i \in B_i \text{ and } v_i \in SC_{i+1}^+(v_{i+2}) \} \]

else //Case 1.

\[ M = \{ b'_i | b'_i \in B_i \text{ and } v_i \notin SC_{i+1}^+(v_{i+2}) \} \]

end if

\( B_i = B_i - M \).

Send \( M \) to \( v_{i+2} \).

Receive blocks from \( v_{i-2} \) and add these blocks to \( B_i \).

end if

**Phase SP_{i}^+:** /\( l = (d-2), (d-3), \ldots, 0 \).

if \( v_i \in V_t \) then

\[ M = \{ b'_j | b'_j \in B_j \text{ and } v_j \in SC_{i}^+(v_{i+2}) \} \]

\( B_i = B_i - M \).

Send \( M \) to \( v_{i+2} \).

Receive blocks from \( v_{i-2} \) and add these blocks to \( B_i \).

end if

The correctness of the routing algorithm can be seen as follows: Case 1 of the algorithm \( GP_{i}^+ \) describes that a block \( b'_i \) will stay in \( v_i \) if \( v_i \in SC_{i+1}^+(v_{i+2}) \) or be moved to \( v_{i+2} \) for joining later phases. Case 2 of the algorithm \( GP_{i}^+ \) describes that \( b'_i \) will be moved from \( v_i \) to \( v_{i+2} \) if \( v_i \in SC_{i+1}^+(v_{i+2}) \). Otherwise, \( b'_i \) stays in \( v_i \) because \( v_i \in SC_{i}^+(v_{i}) \) (Fig. 2b) or it will join later phases. Therefore, after the gathering phases
GP_0^+ \rightarrow GP_1^+ \rightarrow \ldots \rightarrow GP_{d-2}^+ are completed, the block b'_s will be located in some \( v_m \) such that \( v_i \in SC(V_m) \) and \( v_m \in V_0 \cup V_h \), for some \( h, 0 \leq h \leq (d - 1) \). Then, the scattering phases \( SP_0^+ \rightarrow SP_1^+ \rightarrow \ldots \rightarrow SP_{d-3}^+ \) will ensure that \( b'_s \) can reach its destination \( v_i \).

2.4 Overlapping Positive Phases and Negative Phases

We have derived the routing on the positive gather-scatter tree; routing on the negative tree can be similarly obtained. To perform complete exchange, one naive solution is to sequentially perform the positive phases followed by the negative phases. Apparently, this is inefficient as half of the links will be unused in each phase. A better solution is to overlap positive phases and negative phases:

\[
(GP_0^+ \cup GP_0^-) \rightarrow (GP_1^+ \cup GP_1^-) \rightarrow \ldots \rightarrow (GP_{d-2}^+ \cup GP_{d-2}^-) \rightarrow (SP_{d-2}^+ \cup SP_{d-2}^-) \rightarrow \ldots \rightarrow (SP_0^+ \cup SP_0^-).
\]

However, problems may arise because some nodes may need to send/receive more than one message in one phase, thus violating the one-port model. Below we show how to modify our algorithm to solve this problem.

First, we shift the communication patterns in all negative phases, except \( GP_0^- \) and \( SP_0^- \), along the positive direction by one position. That is, we redefine the following negative phases:

\[
GP_i^- = SP_i^- = \{v_1 \rightarrow v_{i-1} \mid (i - 1) \mod 2^l = 0\},
\]

\[1 \leq l \leq (d - 2).
\]

This will relieve the necessity for a node to send/receive more than one message in phase \( GP_i^+ \cup GP_i^- \) and phase \( (SP_i^+ \cup SP_i^-) \), \( 1 \leq l \leq (d - 2) \). For example, see the second to fifth phases in Fig. 4. Note that \( v_i \) and \( \hat{v}_i \) for the negative tree need to be adjusted accordingly to adapt to those changes.

However, since all transmissions of \( GP_0^+ \), \( GP_0^- \), \( SP_0^+ \), and \( SP_0^- \) are of distance one, the above shifting technique does not help to satisfy the one-port constraint. Therefore, we redefine \( GP_0^+ \), \( GP_0^- \), \( SP_0^+ \), and \( SP_0^- \) by removing some transmissions from them as follows:

\[
GP_0^+ = \{v_i \rightarrow v_{i+1} \mid i \mod 2 = 1\},
\]

\[
SP_0^+ = \{v_i \rightarrow v_{i+1} \mid i \mod 2 = 0\},
\]

\[
GP_0^- = \{v_i \rightarrow v_{i-1} \mid i \mod 2 = 0\},
\]

\[
SP_0^- = \{v_i \rightarrow v_{i-1} \mid i \mod 2 = 1\}.
\]

Now, \( GP_0^+ \cup GP_0^- \), as well as \( SP_0^+ \cup SP_0^- \), will conform to the 1-port constraint, as shown in Fig. 4 (the first and last phases).

With these changes, we need to modify the routing of some \( b'_s \), \( \text{dist}(v_s, v_i) \leq 2 \), too. Taking source \( v_0 \) as an example, blocks \( b_1^0 \) and \( b_2^0 \) that would have been sent in the original \( GP_0^+ \) will be left undelivered (since \( v_0 \rightarrow v_1 \) is removed) and, thus, kept in \( v_0 \). Fortunately, this can be taken care of by delivering \( b_1^0 \) in \( v_0 \rightarrow v_1 \) of \( SP_0^+ \) and delivering \( b_2^0 \) in \( v_0 \rightarrow v_2 \) of \( GP_0^+ \). Similarly, the block \( b_1^0 \) of \( v_1 \) that would have been sent in the original \( SP_0^- \) will be undeliverable since \( v_1 \rightarrow v_2 \) is removed. Still, this can be
solved by sending $b_l^i$ in $v_i \rightarrow v_2$ of $G_P^+$. We can accommodate these changes in the routing rules of $G_P^+$, $G_T^+$, $S_P^+$, and $S_P^-$, $l = 0, 1$. All other phases remain the same.

2.5 Performance Analysis

In the following, we analyze the total latency of our complete exchange scheme on a ring of length $2^d$ with message block size $b$, startup time $t_s$, and transmission time $t_x$. Lemmas 2 to 6 show the communication latency of the original positive phases (i.e., without considering the modification in Section 2.4). Finally, Theorem 1 gives the total communication latency incurred by the phases with the changes in Section 2.4. The proofs of Lemmas 2 to 6 and Theorem 1 can be found in the appendix.

Lemma 2. The latency of $G_P^+$, $0 \leq l \leq d-3, d \geq 3$, is

$$T_{1D,G_P^+}(d, b) = t_s + \max \left\{ \left( 2^{d+l-1} - 5 \cdot 2^{l-1} + 3 \cdot 2^{l-7} \right) \cdot b \cdot t_x \right\}.$$

(1)

Lemma 3. The latency of $G_P^+$, $d \geq 3$ is

$$T_{1D,G_P^+}(d, b) = t_s + (2^{d-6} + 3 \cdot 2^{d-3}) \cdot b \cdot t_x.$$

Lemma 4. The latency of $S_P^+$, $d \geq 3$, is

$$T_{1D,S_P^+}(d, b) = t_s + b \cdot t_x.$$

Lemma 5. The latency of $S_P^+$, $0 \leq l \leq d-3, d \geq 3$, is

$$T_{1D,S_P^+}(d, b) = t_s + \max \left\{ \left( 2^{d+l-1} - 5 \cdot 2^{l-1} + 3 \cdot 2^{l-7} \right) \cdot b \cdot t_x \right\}.$$

(2)

From 2 and 5, we find the interesting coincidence that $T_{1D,G_P^+}(d, b) = T_{1D,S_P^+}(d, b), 0 \leq l \leq (d-3)$. Also, there are two terms in the max function in (1) or (2). The following corollary resolves the max function when $d \geq 4$.

Corollary 1. The latency of a positive phase $G_P^+$ or $S_P^+$, $0 \leq l \leq d-4, d \geq 4$, is

$$T_{1D,G_P^+}(d, b) = T_{1D,S_P^+}(d, b) = t_s + (2^{d+l-1} - 5 \cdot 2^{l-1} + 3 \cdot 2^{l-1}) \cdot b \cdot t_x,$$

and, if $l = d-3$, is

$$T_{1D,G_P^+}(d, b) = T_{1D,S_P^+}(d, b) = t_s + \left\{ \begin{array}{ll}
3 \cdot 2^{d-7} + 3 \cdot 2^{d-4} \cdot b \cdot t_x & \text{if } 3 \leq d \leq 5, \\
(7 \cdot 2^{d-8}) \cdot b \cdot t_x & \text{if } d \geq 6. 
\end{array} \right.$$
Proof. By Theorem 1, the cost of the X-stage is \(T_{1D}(d, 2^d \cdot b)\) as the data to be forwarded from a node to any other node contains \(2^d\) blocks. This is the same for the Y-stage. \(\square\)

3.2 A Network-Partitioning Approach: Algorithm T4

The obvious deficiency of algorithm T1 is that transmissions always happen along either the X dimension or Y dimension, but not both. The implication here is that at least half of communication bandwidth is waste. To fix this problem, we propose a new scheme called T4, which is so named because four copies of T1 will be running simultaneously.

The idea is similar to the network-partitioning approach proposed in [34], [35]. We will construct four logical tori, \(P_{i,j}\), \(0 \leq i, j < 2\), each of size \(\frac{2^d}{2} \times \frac{2^d}{2}\). The logical torus \(P_{i,j}\) consists of nodes 
\[
\{v_{(x,y)} | (x \mod \sqrt{4}) = i \text{ and } (y \mod \sqrt{4}) = j\}.
\]

In \(P_{i,j}\), a node \(v_{(x,y)}\) is considered to have a logical link (which is physically dilated by two) to each of nodes \(v_{(x+2,y)}\) and \(v_{(x,y+2)}\). For instance, Fig. 5 shows four logical tori in an \(8 \times 8\) torus. However, communication can be performed in a dilated torus as fast as it can in an ordinary torus due to the distance-insensitive property of wormhole routing. Two important properties offered by such logical partitioning are:

**P1.** The four logical tori \(P_{i,j}\), \(0 \leq i, j < 2\), are node-disjoint.

**P2.** Tori \(P_{0,0}\) and \(P_{1,1}\) are link-disjoint, and \(P_{1,0}\) and \(P_{0,1}\) are link-disjoint.

Next, we need to schedule complete exchange on these four logical tori. Property P1 guarantees that we can freely use these tori without violating the 1-port constraint. P2 guarantees that we can simultaneously run algorithm T1 on tori \(P_{0,0}\) and \(P_{1,1}\) without any link contention. We observe that more saving can be obtained by running algorithm T1 on \(P_{1,0}\) and \(P_{0,1}\) by swapping the execution order to first running Y-stage and then X-stage. The communication directions are summarized in Table 1.

Note that there is no link contention among all these four tori. Also note that, although the above scheduling does utilize all links in every phase, blocks may not reach some of their destinations since the logical tori are node-disjoint. So, some preparation phases shown below are necessary. We schedule every node (say, \(v_{(x,y)} \in P_{i,j}\)) to forward its blocks aimed at nodes in the other three tori \(P_{i\pm 1,j}, P_{i,j\pm 1}\), and \(P_{i\pm 1,j\pm 1}\) (note that, here, “mod 2” is necessary for subscripts larger than one) before performing the above two stages. This can be done in two phases:

**Pre1.** Node \(v_{(x,y)}\) sends to \(v_{(x+1,y)}\) all blocks aimed at \(P_{i+1,j}\) and \(P_{i+1,j+1}\).

**Pre2.** Node \(v_{(x,y)}\) sends to \(v_{(x,y+1)}\) all blocks, together with the blocks received from \(v_{(x-1,y)}\) (in Pre1), aimed at \(P_{i,j}\).

The result is that each \(v_{(x,y)} \in P_{i,j}\) has collected blocks from \(v_{(x-1,y)}, v_{(x,y-1)}\), and \(v_{(x-1,y-1)}\) aimed at nodes in \(P_{i,j}\) and will deliver these blocks in place of these three nodes. In both phases Pre1 and Pre2, \(n^2/2\) blocks are sent.

**Theorem 2.** On a \(2^d \times 2^d\) torus, \(d \geq 4\), the communication latency incurred by algorithm T4 is
\[
T_{2D,T4}(d, b) = 2t_x + 2^d \cdot b \cdot t_x + T_{2D,T1}(d - 1, 4b).
\]

Proof. The first two terms are incurred by the two preprocessing phases. The last term is by algorithm T1, which is run concurrently on all four logical tori (each of size \(2^{d-1} \times 2^{d-1}\)). As each node needs to represent three other neighbors, the latency is \(T_{2D,T1}(d - 1, 4b)\). \(\square\)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>The Scheduling of Communication Directions of Algorithm T4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_{0,0})</td>
<td>(P_{1,1})</td>
</tr>
<tr>
<td>Stage 1</td>
<td>X-stage</td>
</tr>
<tr>
<td>Stage 2</td>
<td>Y-stage</td>
</tr>
</tbody>
</table>
4 EXTENSION TO 3D TORI

In the following, we show how our approaches is extended for a 3D torus. Nodes in the torus will be denoted as \( v_{(x,y,z)} \), \( 0 \leq x, y, z \leq n - 1 \).

Similar to Section 3, we also develop a naive 3-stage scheme, called C1, by first performing an X-stage on each ring along the \( x \)-axis, then a Y-stage on each ring along the \( y \)-axis, and then a Z-stage on each ring along the \( z \)-axis. Apparently, each stage will take time \( T_{1D}(d, n^2 \cdot b) \), so the total time is \( 3 \cdot T_{1D}(d, n^2 \cdot b) \).

As before, to better utilize the communication bandwidth, we need to partition the network into a number of smaller 3D logical tori. The notions behind constructing these logical tori are as follows: First, as there are three stages \( X, Y, \) and \( Z \) (in the naive scheme), nodes along each axis should be divided into at least three logical tori to fully utilize all communication links. Second, these logical tori should be cubic (of equal sizes along all dimensions); therefore, no matter which stage \( X, Y, \) or \( Z \) a torus is scheduled to execute, the communication time is about the same (thus, no torus needs to wait for others to complete).

Based on these observations, one possibility is to partition each axis into four logical rings. Thus, we define \( 4^3 \) dilation-4 logical tori \( C_{i,j,k} \), \( 0 \leq i, j, k < 4 \), such that \( C_{i,j,k} \) consists of nodes

\[ \{v_{(x,y,z)} \mid (x \mod 4) = i, (y \mod 4) = j, \text{and} \ (z \mod 4) = k \}. \]

In \( C_{i,j,k} \), node \( v_{(x,y,z)} \) is considered to be logically adjacent to six nodes \( v_{(x\pm4,y,z)}, \ v_{(x,y\pm4,z)}, \) and \( v_{(x,y,z\pm4)} \). The logical connection is physically dilated by four links. A property similar to \( P1 \) in Section 3.2 is:

\( P1' \): The 64 logical tori \( C_{i,j,k} \), \( 0 \leq i, j, k < 4 \), are mutually node-disjoint.

However, some logical tori do share common links. So, we classify the tori, according to their link sets, into four groups, \( G_s \), \( 0 \leq s \leq 3 \),

\[ G_s = \{C_{i,j,k} \mid (i + j + k) \mod 4 = s \}. \]

Each \( G_s \) contains 16 logical tori. A property similar to \( P2 \) in Section 3.2 is:

\( P2' \): All 16 logical tori in each \( G_s \) are link-disjoint.

Similar to the development in Section 3.2, we can schedule any communication on these 64 logical tori without violating the 1-port constraint (\( P1' \)). Simultaneously performing any communication stage (\( X \)-, \( Y \)-, or \( Z \)-stage) is free from contention in all tori of \( G_s \) (\( P2' \)). One possible arrangement is shown in Table 2. Intuitively, the scheduling of each stage is obtained by cyclically shifting that in the previous stage. Every torus group will perform one X-stage, one Y-stage, and one Z-stage in some order. One nice property is that all \( n^2 \) axes along each dimension are busy at each stage. The scheme is named C64 because it is featured by having 64 logical tori running C1 simultaneously.

---

**TABLE 2**

<table>
<thead>
<tr>
<th>Stages</th>
<th>( G_0 )</th>
<th>( G_1 )</th>
<th>( G_2 )</th>
<th>( G_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 1</td>
<td>X-stage</td>
<td>Y-stage</td>
<td>Z-stage</td>
<td></td>
</tr>
<tr>
<td>Stage 2</td>
<td>X-stage</td>
<td>Y-stage</td>
<td>Z-stage</td>
<td></td>
</tr>
<tr>
<td>Stage 3</td>
<td>Z-stage</td>
<td>X-stage</td>
<td>Y-stage</td>
<td></td>
</tr>
<tr>
<td>Stage 4</td>
<td>Y-stage</td>
<td>Z-stage</td>
<td>X-stage</td>
<td></td>
</tr>
</tbody>
</table>

---

Fig. 6. Illustration of communication pattern in stage 1 in a 3D torus.
Fig. 7. The first three preparation phases.

Fig. 6 demonstrates the communication in Stage 1 following Table 2 (for clarity, only a portion of the torus is shown). Observe that all physical links are utilized. The similar phenomena will occur in other stages, too.

It remains to describe the preparation phases. We can let each node \( v_{(x,y,z)} \) forward \( n^2/64 \) blocks to each \( v_{(x+\delta_x,y+\delta_y,z+\delta_z)} \), \( 0 \leq \delta_x, \delta_y, \delta_z \leq 3 \). The latter nodes will be responsible for delivering the received blocks to other nodes in their logical tori. It suffices to use nine phases as follows:

1. three phases each of the pattern \( v_{(x,y,z)} \rightarrow v_{(x+1,y,z)} \),
2. three phases each of the pattern \( v_{(x,y,z)} \rightarrow v_{(x,y+1,z)} \),
   and
3. three phases each of the pattern \( v_{(x,y,z)} \rightarrow v_{(x,y,z+1)} \).

The three phases in 1 are illustrated in Fig. 7, where the nodes in the gray area are the target to which blocks are gathered. The three phases in 2 and 3, the numbers of blocks sent in them will be exactly the same as that in 1.

Theorem 3. On a \( 2^d \times 2^d \times 2^d \) torus, \( d \geq 5 \), the communication latency incurred by our algorithm C\( 64 \) is

\[
T_{3D}(d, b) = 9t_s + 9 \cdot 2^{3d-1} \cdot b \cdot t_x + 4 \cdot T_{1D}(d - 2, 64 \cdot 2^{2(d-2)} \cdot b).
\]

Proof. The preprocessing cost is

\[
3 \cdot (3t_s + (48 + 32 + 16)\frac{2^{2d}}{M} \cdot b \cdot t_x),
\]

which gives the first two terms. The last term is from the cost of complete exchange on a ring multiplied by four stages. \( \square \)

5 Extension to Nonsquare, Non-Power-of-2 Tori

Up to this point, it seems that our approach can only be applied to tori whose side lengths are equal and power-of-2. Recall that, in Section 2, we developed a (perfect) gather-scare tree on a ring of length \( n = 2^d \). In fact, if some irregularity is allowed, on a ring of any size \( n \), a positive gather-scare tree can be obtained by slightly modifying Definition 1 as follows:

\[
GP_i^+ = SP_i^+ = \{ v_{i \rightarrow v_j} | i \mod 2^j = 0 \},
\]

where \( j = \begin{cases} 
  i + 2^j & \text{if } i + 2^j < n, \\
  0 & \text{otherwise.}
\end{cases} \)

In \( GP_i^+ = SP_i^+ \), whenever the destination node \( v_{i+2^j} \) does not exist, we “unwrap” the destination to node \( v_0 \).

Let \( d = \lfloor \log n \rfloor \). The positive gather-scare tree is still defined based on the \( 2(d-1) \) phases:

\[
GP_0^+ \rightarrow GP_1^+ \rightarrow \ldots \rightarrow GP_d^+ \rightarrow SP_d^+ \rightarrow SP_{d-1}^+ \rightarrow \ldots \rightarrow SP_0^+.
\]

For instance, Fig. 8 shows the positive gather-scare trees on rings of lengths \( n = 10 \) and \( n = 13 \).

The gathering coverage and scattering coverage should be changed accordingly, for instance, the gathering coverage \( GC_3^+ (v_0) = \{ v_0, v_9, v_{12}, \ldots, v_3 \} \) in Fig. 8a and the scattering coverage \( SC_3^+ (v_{12}) = \{ v_{12}, v_9, v_{11}, \ldots, v_3 \} \) in Fig. 8b. A reachability property similar to Lemma 1 will still hold true. We conjecture that the communication time of our complete exchange on a ring of length \( n \) may be upper-bounded by \( T_{1D}(\lfloor \log n \rfloor, b) \). However, the problem of deriving the exact formula of the performance of complete exchange on a non-power-of-2 ring is an open problem to us.

With the availability of complete exchange on a ring of any size, the extension to higher-dimensional, non-power-of-2 tori can be obtained following the line of development in earlier sections. It is also straightforward to extend our scheme to a nonsquare torus since we take a dimension-by-dimension approach.

6 Performance Comparison

In this section, we compare our algorithms (T4 and C64) against those by [25] (T-2D1, T-2D2, T-3D1, and T-3D2) and [32] (DP-2D and DP-3D). The following lemma will be used for analyzing these schemes.

Lemma 8. To perform complete exchange on a 2D \( n \times n \) torus (resp., 3D \( n \times n \times n \) torus), a lower bound on the startup time is \( \log(n^2) t_s \) (resp., \( \log(n^3) t_s \)) and a lower bound on the transmission time is \( n^2 b \cdot t_x \) (resp., \( n^3 b \cdot t_x \)).

2. In [26], algorithms T-2D1 and T-3D1 are referred to as T2D and T3D, respectively.
Proof. The lower bounds on the startup time are obtained by taking logarithm of the network size. The bounds on transmission time are established in [32]. □

Table 3 shows the startup and transmission costs of our and others’ algorithms on a 2D \(2^d \times 2^d\) torus. In terms of startup cost, both T-2D1 and our T4 have an order of \(O(d)\), while T-2D2 and DP-2D have an exponential order of \(O(2^d)\). Our T4 incurs a slightly higher startup time, but, as will be shown later, this will be offset by the transmission time, which is relatively more significant. In terms of transmission cost, T-2D1, T-2D2, and DP-2D are about \(\frac{9}{2}, 3,\) and \(2\) times the lower bound, respectively. Our T4 requires the least transmission time, about \(\frac{65}{48} \approx 1.35\) times the lower bound.

Table 4 shows the startup and transmission costs of our and others’ algorithms on a 3D \(2^d \times 2^d \times 2^d\) torus. In terms of transmission cost, T-3D1, T-3D2, and DP-3D are about \(\frac{10}{1}, 3,\) and \(2\) times the lower bound, respectively. Our C64 requires transmission time of about \(\frac{65}{48} \approx 1.35\) times the lower bound.

From Table 3 and Table 4, we establish in Table 5 the theoretical speedups on transmission time of our T4 over T-2D1, T-2D2, DP-2D, and our C64 over T-3D1, T-3D2, DP-3D, when \(d\) approaches infinity. These speedups can be used as a reference in our following analyses when we compare other algorithms to our algorithms.

Next, we further study the impact of ratio \(\frac{ts}{tx}\). We plot Fig. 9 using different ratios of \(\frac{ts}{tx}\), namely, \(2,500, 500, 100,\) and \(1\) at different torus sizes. The plots are obtained by dividing the latency of other algorithms by that of ours (thus, a speedup value larger than 1 indicates the advantage of our algorithm). The largest ratio, 2,500, is chosen for the following reason: In Intel Paragon, \(t_s = 216\text{us}\) and \(t_x = 0.0226\text{us/byte}\) [12]; letting \(b = 4\) we have \(\frac{ts}{tx} \approx 2,500\). We observe that, in most cases, the speedup is larger than 1 for all ratios of \(\frac{ts}{tx}\). Only when \(d\) is small (\(d = 4 \sim 6\) in Fig. 9a and \(d = 4 \sim 5\) in Fig. 9b, c, d) the speedups are less significant because our algorithms take more steps (startup times) than others. After \(d \geq 6\), the speedups will approach the values in Table 5 because the transmission costs will become the dominating factor.

In Fig. 10, we take a closer look at the relationship between the speedup and the message block size \(b\) by fixing the ratio \(\frac{ts}{tx}\) to 10,000, 2,000, and 400. Fig. 10a shows the speedups obtained by our T4 when \(d = 4\) (a \(16 \times 16\) torus). The speedup of T4 over T-2D1 is less significant when \(b\) is small (\(b = 4 \sim 16\)) because T-2D1 has the lowest startup cost. Thus, a higher ratio of \(\frac{ts}{tx}\) will lead to a lower speedup. When \(b\) gets larger, the transmission time will dominate the overall cost, and the speedup will reach a stable value of 1.89 for T-2D1/T4, 1.33 for T-2D2/T4, and a stable value of 1.11 for DP-2D/T4. Fig. 10b shows a similar trend when the network size is \(d = 5\) (a \(32 \times 32\) torus), but with a higher stable speedup value of 2.94 for T-2D1/T4, 1.81 for T-2D2/T4, and of 1.36 for DP-2D/T4. In Fig. 10c, the

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Startup time ((\times t_s))</th>
<th>Transmission time ( (\times b \cdot t_x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-2D1</td>
<td>((d \geq 4)) (3(d-1))</td>
<td>(9 \cdot 2^{3d-4} + (d^2 - 5d + 3)2^{2d-1})</td>
</tr>
<tr>
<td>T-2D2</td>
<td>((d \geq 4)) (3 \cdot 2^{d-2})</td>
<td>(3 \cdot 2^{3(d-1)})</td>
</tr>
<tr>
<td>DP-2D</td>
<td>((2^{d-1} + 2))</td>
<td>(2^{2d} + 2^{2d})</td>
</tr>
<tr>
<td>T4</td>
<td>((d = 4)) (4d - 6)</td>
<td>(\frac{1}{3} \cdot \frac{31}{32} 2^{3d-1} + 5 \cdot 2^{2d-2} + \frac{3}{2} 2^{d+2})</td>
</tr>
<tr>
<td></td>
<td>((5 \leq d \leq 6)) (4d - 6)</td>
<td>(\frac{1}{3} \cdot \frac{31}{32} 2^{3d-1} + 5 \cdot 2^{2d-2} + \frac{3}{2} 2^{d+2})</td>
</tr>
<tr>
<td></td>
<td>((d \geq 7)) (4d - 6)</td>
<td>(\frac{1}{3} \cdot \frac{65}{64} 2^{3d-1} + 2^{2d-1} + \frac{5}{2} 2^{d+2})</td>
</tr>
</tbody>
</table>

TABLE 3
Comparison of Startup and Transmission Costs of Complete Exchange Schemes on a 2D \(2^d \times 2^d\) Torus
speedup approaches 3.47 for T-2D1/T4, 2.05 for T-2D2/T4, and approaches 1.46 for DP-2D/T4 when \( b \) increases. Note that the speedup is slightly going down for T-2D2/T4 and DP-2D/T4 when \( b \) is small (at \( b \approx 4 \)) in Fig. 10c. This is because both T-2D2 and DP-2D have a higher \( O_{\text{startup}} \) costs and, thus, when \( d \approx 6 \), the save on startup cost is a dominating factor. When \( b \) gets larger, the speedup is flattened out because the transmission time becomes the dominating factor again.

To study the speedups of algorithm C64, we show only the cases of \( t_s/b = 2,500 \) and 1 in Fig. 11a and b, respectively, because the trend under different ratios of \( t_s/b \) are similar. Our algorithm C64 can provide significant speedups in most cases.

Finally, we comment on possible inclusion of synchronization and local data movement costs into the performance evaluation. Suppose a global barrier is inserted between each two consecutive phases to synchronize the communication. Since the cost of a barrier operation is fixed for a given torus, a convenient way is to regard this cost as a part of the startup cost \( t_s \). The local data movement cost refers to the costs of disassembling and reassembling a message to be delivered on a link. This is proportional to the size of the message. Since we already count the size of the message into the communication cost, a convenient way is to regard the local data movement cost per byte as a part of the transmission cost \( t_x \). To summarize, our timing model can reasonably approximate the communication latency as well as the barrier and local data movement costs if we can properly determine the values of \( t_s \) and \( t_x \). This is why we used a wide range of \( t_s/b \) in the above comparison.

### 7. Concerns of Synchronization

In previous analyses, we have assumed that the communication phases happen perfectly synchronized in a step-wise manner, which is not necessarily true. Consider Fig. 12. After \( v_0 \rightarrow v_2 \) is finished in \( GP_{d-1}^+ \), \( v_2 \) does not participate in any of \( GP_d^+ \) and \( SP_d^+ \), so it will proceed to carry out \( v_2 \rightarrow v_4 \) that is supposed to be performed later in \( SP_d^+ \). This transmission \( v_2 \rightarrow v_4 \) may contend with \( v_0 \rightarrow v_2 \) in \( GP_{d-2}^+ \) and even \( v_0 \rightarrow v_3 \) in \( SP_{d-1}^+ \) for the physical links between \( v_2 \) and \( v_4 \). Such contention may disrupt the transmissions in the gather-scatter tree and prolong the overall latency. Below, we discuss three possible approaches, which require different levels of hardware support, to synchronize these phases in our gather-scatter-tree schemes.

#### 7.1 Global Barrier

Apparently, we can add barriers into the following sequence of phases

\[
GP_0^+ \rightarrow GP_1^+ \rightarrow \ldots \rightarrow GP_{d-2}^+ \rightarrow SP_{d-2}^+ \rightarrow SP_{d-3}^+ \rightarrow \ldots \rightarrow SP_0^+.
\]

This requires \( 2d - 3 \) barriers (counting the arrows). However, after careful examination, the two sequences \( GP_{d-2}^+ \rightarrow SP_{d-2}^+ \) and \( SP_1^+ \rightarrow SP_0^+ \) are self-synchronized and, thus, the corresponding barriers are unnecessary. So, only \( 2d - 5 \) global barriers are required. This approach is appropriate for systems that support hardware barrier synchronization.
7.2 Local Synchronization

The previous approach has assumed that efficient hardware-supported barrier synchronization is available. If not so, software-implemented global barriers may be used. However, software barriers will be much more costly. Fortunately, as shown in Fig. 12, these global barriers can be replaced by local synchronization between some pairs of nodes. For instance, we only require that \( v_0 \) wait for a signal from \( v_4 \) after \( GP_1^+ \) to prevent \( v_0 \rightarrow v_4 \) of \( GP_2^+ \) from contending with \( v_2 \rightarrow v_4 \) of \( GP_1^+ \). Similarly, we only require that \( v_2 \) wait for a signal from \( v_0 \) after \( SP_2^+ \) to prevent \( v_2 \rightarrow v_4 \) of \( SP_1^+ \) from starting before \( v_0 \rightarrow v_4 \) of \( SP_2^+ \) has finished.

The above signaling can be easily implemented by sending a null message. Thus, the cost should be much less than that of doing global software barriers. In general, after each \( v_i \rightarrow v_{i+2} \) of \( GP_l^+ \), \( l = 1, \ldots, d-2 \), we need to add a signaling \( v_{i+2} \rightarrow v_i \) for each \( v_i \in V_l \). This will eliminate the contention between \( v_{i+2} \rightarrow v_{i+2} \) of \( GP_{l-1}^+ \) and \( v_i \rightarrow v_{i+2} \) of \( GP_l^+ \). Also, after each \( v_i \rightarrow v_{i+2} \) of \( SP_l^+ \), \( l = d-2, \ldots, 2 \), we need to add a signaling \( v_i \rightarrow v_{i+2} \) for each \( v_i \in V_l \). This will eliminate the contention between \( v_i \rightarrow v_{i+2} \) of \( SP_l^+ \) and \( v_{i+2} \rightarrow v_{i+2} \) of \( SP_{l-1}^+ \).

In total, \( 2d - 5 \) times of local synchronization will be used.

7.3 Systems which Support Prioritized Messages

Hardware support of prioritized message delivery [24] in wormhole networks has received some attention recently. In such systems, messages with higher priorities can preempt the network resources (input buffer and output channel) of lower priority messages. In [24], a throttle mechanism is proposed to preserve input buffers to ensure that, when the header flit with a higher priority arrives, the wormhole router can accept this header and, then, this header (and, thus, those flits following this header) can preempt the output channel if it is currently used by flits with a lower priority. After the tail flit of the higher priority worm leaves the router, the preempted channel is returned to the original worm.

If the above prioritized message delivery is supported by the underlying system, we may assign higher priorities to messages in earlier phases to prevent them from being blocked by messages of later phases. In this way, the overall latency will not be degraded even if some transmissions of posterior phases start before prior phases. To support our schemes, the system should provide at least a number of priority levels equal to the number of phases.
Fig. 10. Speedup of complete exchange obtained by our algorithm T4 vs. block size \( b \) on a \( 2^d \times 2^d \) torus with \( \frac{t_1}{t_2} = 10,000 \), \( \frac{t_1}{t_2} = 2,000 \), and \( \frac{t_1}{t_2} = 400 \). (a) \( d = 4 \). (b) \( d = 5 \). (c) \( d = 6 \).
8 CONCLUSIONS

In this paper, we have presented a systematic solution to perform complete exchange in a torus network using wormhole routing. The solution can be used on nonsquare, non-power-of-2, any-dimensional tori, and this is the first result known to us with such generality in the literature. Interesting techniques used in this paper include the gather-scatter tree structure, network-partitioning approach, and dimension-by-dimension strategy to optimize the startup and transmission costs subject to wormhole routing.

Performance-wise, when the torus is square, our 2D and 3D schemes incur asymptotically optimal startup and transmission time (about $2\frac{1}{3}$ times the startup lower bound and both $1.35$ times the transmission lower bound for 2D and 3D schemes, respectively). Numerical evaluation has shown significant speedup of these schemes over existing schemes at various communication parameters. Future research may be on reducing the constants associated with the startup and transmission complexities.

APPENDIX

For convenience, we define $\Phi^+ = 2^{d-1} + 1$. Intuitively, this is the number of destination nodes a node needs to cover in the positive tree plus one (by “plus one,” we imagine that the node itself is also a destination).

Proof of Lemma 2. Consider the transmission $v_i \rightarrow v_i + 2^l$ of $GP_i^+$. There are two cases (recall the discussion in Section 2.2).

Case 1: $v_i \in \hat{V}_{l+1}$. Since $v_i \in V_{l'}$, blocks in $B_l$ are gathered from $2^{l'}$ nodes in $GC_i^+(v_i)$. According to the algorithm, any $b_{l'} \in B_l$ that $v_i \notin SC_{l+1}^+(v_i)$ will be sent (refer to Fig. 13a). This implies that any block in $B_l$ whose destination is farther than $v_i + 2^{l'+1}$ will be sent. Thus, for first source $v_{l-(2^l-1)} \in GC_i^+(v_i)$, blocks $b_{l-(2^l-1)}$ will be sent in $v_i \rightarrow v_i + 2^l$, where $v_{l-(2^l-1)}$ is the farthest destination of the source in the positive tree. This includes $3 \cdot 2^l + 1$ blocks.

Summing these together, node $v_i$ needs to send $2^l$ blocks.

Fig. 11. Speedup of complete exchange obtained by our algorithm C64 at different values of $d$. (a) $\frac{1}{\sqrt{d}} = 2.500$. (b) $\frac{1}{\sqrt{d}} = 1$.

Fig. 12. Additional barriers/synchronization in the positive gather-scatter tree of a 16-node ring to avoid link contention.
blocks. Note that the above analysis is correct since all terms in $\Phi$ are positive numbers.

**Case 2:** $v_i \in V_{l+1}$. First, consider $1 \leq l \leq d - 3$. According to the algorithm, any $b'_l \in B_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent to $v_{i+l+2}$. However, there is some intersection between $SC^+_{l+1}(v_{i+l+2})$ and $SC^+_{l+1}(v_{i+l+2})$ (see the illustration in Fig. 13b). So, the analysis depends on the location of $v_i$.

1. $v_i \in GC^+_{d-3}(v_{i+2l-1})$ (first half of $GC^+_{d-2}(v_i)$): Any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent in $v_i \rightarrow v_{i+l+2}$. So, in total, $2^{l+1} - 2^{l+1} = 2^{l+1}$ blocks ($b'_l \in 2^{l+1}$) will be sent.

2. $v_i \in GC^+_{l+1}(v_i)$ (second half of $GC^+_{l}(v_i)$): Any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2}) \cap SC^+_{l+1}(v_{i+l+2})$ has been sent in $v_i \rightarrow v_{i+l+2}$ of the previous $GP^+_{d-1}$. So, only the remaining $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2}) - SC^+_{l+1}(v_{i+l+2})$ needs to be sent. Therefore, in total, $2^{l+1} - 3 \cdot 2^{l+1} = 3 \cdot 2^{l+1}$ blocks ($b'_l \in 3 \cdot 2^{l+1}$) will be sent.

As a result, $2^{l+1} - 3 \cdot 2^{l+2} = 7 \cdot 2^{l+2}$ blocks are sent in this case. Note that the above analysis is valid as every $v_i$ considered has one block for the farthest destination $v_{i+3}$ when $1 \leq l \leq d - 3$. As for $l = 0$, exactly two blocks $2^{l+1} + 1$ will be sent.

The lemma then follows by taking the maximum of the costs incurred by the above two cases. Note that, when $l = 0$, the cost of Case 2 (two blocks) is overwhelmed by that of Case 1. \qed

**Proof of Lemma 3.** The proof is similar to that of Lemma 2. Consider the transmission $v_i \rightarrow v_{i+l-1}$ of $GP^+_{l+2}$. As $v_i$ must be in $V_{l-1}$, only Case 2 in the proof of Lemma 2 sustains. According to the algorithm, any $b'_l \in B_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent to $v_{i+l+2}$. However, there is an intersection between $SC^+_{l+1}(v_{i+l+2})$ and $SC^+_{l+1}(v_{i+l+2})$. So, the analysis depends on the location of $v_i$.

1. $v_i \in SC^+_{l+1}(v_{i+2l-1})$ (first half of $SC^+_{l}(v_i)$): Any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent. For source $v_i \rightarrow v_{i+l+2-1}$, any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent. Similar calculation can be done for other sources (e.g., for $v_i \rightarrow (v_{i+l+2-1}+1)$, $\Phi^+ = 2^{d-l} + 2$ blocks will be sent). Summing these together, $2^{d-l} - 2^{d-l} + 1 + j = 2^{d-l} + 3 \cdot 2^{d-l}$ blocks will be sent.

2. $v_i \in GC^+_{l}(v_i)$ (second half of $GC^+_{l+1}(v_i)$): The blocks to be sent are those $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2}) - SC^+_{l+1}(v_{i+l+2})$. For source $v_i \rightarrow v_{i+l+2}$, there are $\Phi^+ = 2^{d-l} + 2$ blocks, i.e., $b'_l \in 2^{d-l}$. Similar calculation can be done for other sources (e.g., for $v_i \rightarrow (v_{i+l+2}+1)$, $\Phi^+ = 2^{d-l} + 2$ blocks will be sent). Summing these together, $2^{d-l} - 2^{d-l} + 1 + j = 2^{d-l} + 3 \cdot 2^{d-l}$ blocks will be sent.

Therefore, we need to send $2(2^{d-l} + 3 \cdot 2^{d-l}) = 2^{d-l} + 3 \cdot 2^{d-l}$ blocks in $v_i \rightarrow v_{i+l+2}$ of $GP^+_{l+2}$. The above analysis is valid for $d \geq 3$. \qed

**Proof of Lemma 4.** Consider the transmission $v_i \rightarrow v_{i+l+2}$ of $ST^+_{d-2}$. Blocks in $B_l$ are gathered from all nodes of $GC^+_{d}$. According to the algorithm, any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent. However, $SC^+_{d-1}(v_{i+l+2}) \subseteq SC^+_{l+1}(v_{i+l+2})$, where the latter is considered in previous $v_i \rightarrow v_{i+l+2}$ of $GP^+_{d-2}$. So, the analysis depends on the location of $v_i$.

1. $v_i \in GC^+_{d-2}(v_{i+2l-1})$ (first half of $GC^+_{d-1}(v_i)$): For the first destination $v_{i+2l-1}$, any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ will be sent. The only block satisfied this condition is $b'_l \in 2^{d-l}$. There is no block to be sent to other destinations in $GP^+_{d-2}(v_{i+l+2})$.

2. $v_i \in GC^+_{d-2}(v_i)$ (second half of $GC^+_{d-1}(v_i)$): In $GP^+_{d-2}$, any $b'_l$ that $v_i \in SC^+_{l+1}(v_{i+l+2})$ has already
been sent to \(v_i + 2^{2l-2}\). Thus, there is no block left to be sent for these sources.

Consequently, one block will be sent in \(v_i \rightarrow v_{i+2^{2l-2}}\) of \(SP_{d-2}^+\). (An instance of this lemma can be observed in Fig. 3, where \(v_0\) is the source and \(v_8\) is the destination.)

**Proof of Lemma 5.** Consider the transmission \(v_i \rightarrow v_{i+2}\) of \(SP_{i}^+\). There are two cases.

**Case 1:** \(v_i \in V_{l+1}(v_{i+2} \in \tilde{V}_{l+1})\). Any \(b'_s \in B_i\) that \(v_i \in SC^+_l(v_{i+2})\) will be sent. As shown in Fig. 14a, the transmission \(v_i \rightarrow v_{i+2}\) in \(GP_{i+1}^+\) has already moved blocks from sources in \(GC^+_i(v_i)\) to \(v_{i+2}\) for destinations in \(SC^+_l(v_{i+2})\). Since \(SC^+_l(v_{i+2}) \subseteq SC^+_l(v_{i+2})\), any \(b'_s\) which remains in \(B_i\) such that \(v_i \in SC^+_l(v_{i+2})\) must have \(v_i\) farther than \(v_{i+2}\). Thus, for the first destination \(v_{i+2}\), \(2^{2l-1} - 2^{2l-1} + 1\) blocks \(b'_s\) will be sent, where \(v_{i+2} \rightarrow (v_{i+2})\) is the farthest source that \(v_{i+2}\) could involve. Similar calculation can be done for other destinations in \(SC^+_l(v_{i+2})\) (e.g., \(v_{i+2} \rightarrow (v_{i+2})\)). Summing these together,

\[
\sum_{j=0}^{2^{2l-1}} (2^{2l-1} - 2^{2l-1} + 1 - j) = 2^{2l-1} - 5 \cdot 2^{2l-1} + 3 \cdot 2^{2l-1}
\]

blocks will be sent. The analysis is valid since all terms in \(\sum\) are positive.

**Case 2:** \(v_i \in \tilde{V}_{l+1}(v_{i+2} \in \tilde{V}_{l+1})\). First, consider \(1 \leq l \leq d - 3\). Since \(v_i \in \tilde{V}_{l+1}\), blocks in \(B_i\) are gathered from \(2^{2l-1}\) nodes \(GC^+_i(v_i)\). Any \(b'_s \in B_i\) that \(v_i \in SC^+_l(v_{i+2})\) will be sent. Some blocks heading for destinations in \(SC^+_l(v_{i+2}) \subseteq SC^+_l(v_{i+2})\) have already been sent in \(GP_{i+1}^+\) (Fig. 14b). So, the analysis depends on the location of \(v_{i+2}\).

1. \(v_i \in SC^+_l(v_{i+2})\) (first half of \(SC^+_l(v_{i+2})\)): Source nodes in \(GC^+_i(v_i)\) already have Case 2 \(v_i \rightarrow v_{i+2}\) in \(GP_{i+1}^+\) move blocks whose destinations include those in \(SC^+_l(v_{i+2})\). Thus, any \(b'_s\) that \(v_i \in GC^+_i(v_i)\) will be sent. That is, \(3 \cdot 2^{2l-1}\) blocks \(b'_s\) will be sent to each \(v_i\). So, in total, \(2^{2l-1} \cdot 3 \cdot 2^{2l-1} = 3 \cdot 2^{2l-2}\) blocks will be sent.
2. \(v_i \in SC^+_l(v_{i+2}+1)\) (second half of \(SC^+_l(v_{i+2})\)): Any \(b'_s\) that \(v_i \in GC^+_i(v_i)\) will be sent. That is, \(2^{2l-1}\) blocks \(b'_s\) will be sent for each \(v_i\). So, in total, \(2^{2l-1} \cdot 2^{2l-1} = 2^{2l}\) blocks are sent.

As a result, \(2^{2l} + 3 \cdot 2^{2l-2} = 7 \cdot 2^{2l-2}\) blocks are sent in a Case 2 transmission of \(SP_{i}^+\). Note that the above analysis is valid as the farthest distance \(dist^+(v_{i+2} \rightarrow (v_{i+2})\) when \(1 \leq l \leq d - 3\). As for \(SP^+_i\), exactly two blocks, i.e., \(b'_s\), will be sent.

The lemma then follows by taking the maximum of the costs incurred by the above two cases. Note that, when \(l = 0\), the cost of Case 2 (two blocks) is overwhelmed by that of Case 1.

**Proof of Corollary 1.** Note that the analyzed result of \(GP^+_i\) (Lemma 2) is identical to that of \(SP^+_i\) (Lemma 5). In each of \(GP^+_i\) or \(SP^+_i\), two message sizes are compared for finding the transmission time of that phase. If \(l \leq d - 4\), the message size of Case 1 is always larger than that of Case 2. Therefore, the transmission time is \((2^{2l-1} - 5 \cdot 2^{2l-1} + 3 \cdot 2^{2l-1}) \cdot b \cdot t_x\). In case of \(l = d - 3\), the message size of Case 1 transmission is larger if \(3 \leq d \leq 5\), but, if \(d \geq 6\), that of Case 2 is larger. Thus, by substituting \(l\) with \(d - 3\), we have \(3 \cdot 2^{2l-7} + 3 \cdot 2^{2l-4}\) for \(3 \leq d \leq 5\) and \(7 \cdot 2^{2l-8} = 7 \cdot 2^{2l-8}\) for \(d \geq 6\).

**Proof of Lemma 6.** The total communication time is

\[
T_{ID}^+(d, b) = \sum_{l=0}^{d-3} T_{1DGP^+_l}(d, b) + T_{1DGP^+_l}(d, b) + T_{1DGP^+_l}(d, b) + T_{1DGP^+_l}(d, b) + T_{1DGP^+_l}(d, b)
\]

\[
+ \sum_{l=0}^{d-3} T_{1DGP^+_l}(d, b).
\]
**Proof of Theorem 1.** Since \( \Phi^+ > \Phi^- = 2^d - 1 \), the latency of a negative phase is smaller than that of a corresponding positive phase. Therefore, we need to consider only the latency of positive part plus the additional cost for overlapping. We adjust the latency for positive phases \( GP_0^+ \) and \( GP_0^- \) according to Section 2.4. Depending on the location of \( v_s \), we have the following four cases:

1. \( v_s \in \mathcal{V}_2 \); \( v_s \Rightarrow v_{s+1} \) of \( GP_0^+ \) and \( v_{s+1} \Rightarrow v_{s+2} \) of \( SP_0^+ \) are removed. Thus, \( b_1^{s+1} \) is sent in \( SP_0^+ \) (Case 1), and \( b_1^{s+1} \) is sent in \( SP_0^- \) (Case 2).
2. \( v_{s+1} \in \mathcal{V}_2 \); \( v_s \Rightarrow v_{s+1} \) of \( SP_0^+ \) is removed. Thus, \( b_1^{s+1} \) is sent in \( GP_0^+ \) (Case 1).
3. \( v_{s+2} \in \mathcal{V}_2 \); \( v_s \Rightarrow v_{s+1} \) of \( GP_0^+ \) and \( v_{s+1} \Rightarrow v_{s+2} \) of \( SP_0^+ \) are removed. Thus, \( b_1^{s+1} \) is sent in \( SP_0^+ \) (Case 1), and \( b_2^{s+1} \) is sent in \( GP_1^+ \) (Case 2).
4. \( v_{s+3} \in \mathcal{V}_2 \); \( v_s \Rightarrow v_{s+1} \) of \( SP_0^+ \) is removed. Thus, \( b_1^{s+1} \) is sent in \( GP_0^+ \) (Case 1).

Because the transmission time of each of \( GP_0^+ \), \( SP_0^+ \), \( GP_0^- \), and \( SP_0^- \) is dominated by Case 1 transmissions, the total transmission time is increased by only two blocks, one added to \( GP_0^+ \) and another to \( SP_0^+ \), for \( d \geq 4 \). If \( d = 3 \), \( GP_1^+ \) is the special case \( GP_{d-2}^+ \) in which every transmission is considered as Case 2. Thus, the additional one block transmitted in \( GP_1^+ \) should be taken into account, where no additional block is transmitted in \( SP_0^+ \) for \( d \geq 4 \). Consequently, three blocks are added to the total transmission time for \( d = 3 \).

**Acknowledgments**

This work was supported in part by the National Science Council of the Republic of China under Grants NSC88-2213-E-008-020, NSC88-2213-E-008-027, and NSC89-2213-E-008-028. This work was supported in part by the National Science Council of the Republic of China under Grants NSC88-2213-E-008-027, and NSC89-2213-E-008-028. A preliminary version of this paper appeared in the Proceedings of the 1997 International Conference on Parallel and Distributed Systems [33].

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