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Information Processing Letters 48 (1993) 237–241

Information Processing Letters

A broadcasting algorithm in star graph interconnection networks

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Communicated by K. Ikeda
Received 20 July 1992
Revised 8 September 1993

Key words: Broadcasting; Interconnection networks; Parallel processing; Star graphs

1. Introduction

Broadcasting is a very important operation widely used in various linear algebra algorithms, database queries, transitive closure algorithms, and linear programming algorithms. The interconnection network must facilitate efficient broadcasting so as to achieve high performance during execution of jobs. Several broadcasting algorithms in star graphs have been presented by [1,3,4] under the assumption of the communication capability that each node sends a message through at most one outgoing link at a time. Their proposed broadcasting algorithms have the same time complexity $O(n \log n)$, where n is the dimension of the star graph. Some nodes will, however, receive the message redundantly.

In this paper, we propose a broadcasting algorithm without message redundancy in the star graphs. The algorithm works by sending some information along with the message that indicates how the algorithm should continue to broadcast the message from the receiving node and how the message sent redundantly should be avoided. A distributed broadcasting algorithm that broadcasts the message to each node exactly once in $2n-3$ time steps is presented here on the basis of the assumption that each node can send a message through all of its outgoing links simultaneously in star graphs. A time step is the actual time needed to send a unit of message from a node to some of its neighboring nodes.

2. A broadcasting algorithm

An n-dimensional star graph [1,2] is defined as a symmetric graph $S_n = (V_n, E_n)$ where V_n is the set of $n!$ nodes and E_n is the set of symmetric edges. Each node $u \in V_n$ is one of the permutations of n distinct symbols. In this paper, assume that n distinct symbols are $1, 2, \dots, n$. Thus $V_3 = \{123, 132, 213, 231, 312, 321\}$. Each edge e belongs to E_n, i.e., two permutations are connected by an edge if and only if one can be reached from another by interchanging its first symbol with any other symbol. Each node clearly has $n-1$ incident edges. Restated, the degree of the S_n is $n-1$. For example, the star graph S_4 is shown in Fig. 1.

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SSDI 0020-0190(93)E0169-K

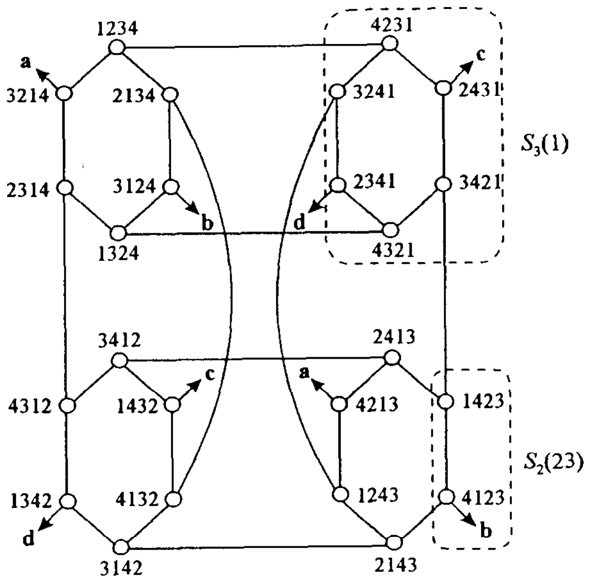

Fig. 1. The star graph S_4 and the substar graphs $S_3(1)$ and $S_2(23)$.

Definition 1 (Generator g_i). Let $a = a_1 a_2 \cdots a_n$ be a label of a node in the star graph S_n. The *generator* g_i on the label a is defined as interchanging the first symbol a_1 with the symbol a_i, for $2 \leqslant i \leqslant n$.

Definition 2 (Substar graph [3]). Let $S_i(a_{i+1}a_{i+2} \cdots a_n)$ denote an i-dimensional *substar graph* of S_n induced by all the nodes with the same last $n-i$ symbols $a_{i+1}a_{i+2} \cdots a_n$, $1 \leqslant i \leqslant n$, where $a_{i+1}a_{i+2} \cdots a_n$ is a permutation of $n-i$ distinct symbols in $\{1, 2, \ldots, n\}$.

Note that the substar graph $S_i(a_{i+1}a_{i+2} \cdots a_n)$ can be regarded as an i-dimensional star graph S_i. For example, $S_3(1)$ and $S_2(23)$ are a three-dimensional star graph and a two-dimensional star graph in the star graph S_4, respectively, as shown in Fig. 1. Due to the fact that star graphs have a hierarchical structure, an n-dimensional star graph S_n can be easily decomposed into n substar graphs S_{n-1}. Therefore, the S_n can be decomposed into n substar graphs S_{n-1} by fixing each of n symbols in one particular position from 2 to n. Each $(n-1)$-dimensional substar graph is isomorphic to S_{n-1}. For example, in S_4, the symbol in the fourth position can be arranged with any symbol. Four substar graphs $S_3(1)$, $S_3(2)$, $S_3(3)$ and $S_3(4)$ can be obtained where the respective symbols in the fourth position are 1, 2, 3, and 4.

The basic idea of our broadcasting algorithm in the star graphs is stated as follows. An n-dimensional star graph S_n can be first decomposed into n substar graphs S_{n-1}. The source node belonging to an substar graph S_{n-1} will be able to send the message to the other $n-1$ substar graphs S_{n-1}. While the source node sends the message to the other $n-1$ substar graphs S_{n-1}, it also performs the broadcasting within the substar graph S_{n-1} which includes the source node.

We illustrate how to broadcast the message in star graphs with the following example. In the star graph S_4, a message of the source node 1234 is considered here to be broadcast to all of other nodes. The star graph S_4 can be decomposed into four substar graphs $S_3(1)$, $S_3(2)$, $S_3(3)$, and $S_3(4)$. If the message is sent to one node of the substar graphs $S_3(i)$, $1 \leqslant i \leqslant 4$, then the broadcasting in each substar graph $S_3(i)$ can be regarded as the broadcasting in the star graph S_3. Because the source node 1234 belongs to $S_3(4)$, the message of the source node would be sent to one node of $S_3(1)$, $S_3(2)$, and $S_3(3)$, respectively. First, we apply the generators g_2, g_3, and g_4 to the node 1234. That is, the message of source node 1234 is sent to the three nodes $g_2(1234) = 2134$, $g_3(1234) = 3214$, and $g_4(1234) = 4231$. Only the node 4231 belongs to $S_3(1)$. The two nodes 2134 and 3214 do not, however, belong to $S_3(2)$ and $S_3(3)$, respectively. Then the node 2134 applies the generators g_3 and g_4 to sending the message to the nodes 3124 and 4132, respectively. The node 3214 applies the generators g_2 and g_4 to sending the message to the nodes 2314 and 4213, respectively. Since the two nodes 4132 and 4213 belong to the substar graphs $S_3(2)$ and $S_3(3)$, respectively, the message can be broadcast to each substar $S_3(i)$, for $1 \leqslant i \leqslant 3$. In Lemma 3, it will be proven to take two time steps for the source node of a star graph S_n to send the source message to each of the $n-1$ substar graphs S_{n-1}. One node of each substar graph after receiving the message can therefore apply the above strategy recursively until each substar graph becomes a one-dimensional star graph. The detailed broadcasting rules will be described in later.

Lemma 3. *It takes two time steps for the source node of a star graph S_n to send the source message to one node of the $n-1$ substar graphs S_{n-1} except the one including the source node.*

Proof. Without loss of generality, assume that the source node is the identity node $12 \cdots n$. The S_n can be decomposed into n substar graphs $S_{n-1}(i)$, for $1 \leqslant i \leqslant n$. First, we apply the generators g_i, for $2 \leqslant i \leqslant n$, to the source node $12 \cdots n$. Thus, we can obtain the nodes

$$v_i = g_i(12 \cdots n) = i2 \cdots (i-1)1(i+1) \cdots n,$$

for $2 \leqslant i \leqslant n$. Only the node

$$v_n = g_n(12 \cdots n) = n23 \cdots (n-1)1$$

belongs to the substar graph $S_{n-1}(1)$. Each node v_i does not, however, belong to the corresponding substar graph $S_{n-1}(i)$, for $2 \leqslant i \leqslant n-1$. Then, we apply the generator g_n to the node v_i so that the node $w_i =$

$$g_n(g_i(12 \cdots n)) = g_n(v_i) = n2 \cdots (i-1)1(i+1) \cdots (n-1)i$$

belongs to the corresponding substar graph $S_{n-1}(i)$, for $2 \leqslant i \leqslant n-1$. Each node w_i clearly belongs to $S_{n-1}(i)$, for $2 \leqslant i \leqslant n-1$. It therefore takes two time steps that the source node of a star graph S_n sends the source message to one node of the $n-1$ substar graphs S_{n-1} except the one including the source node. □

Two routing rules are proposed here for broadcasting in the star graphs in order to avoid sending the message redundantly. After each node has received the message, it can be viewed as either the *source* node of a specific substar graph or not only the *source* node but also the *relay* node of another substar graph. The node performs Rule 1 if it is only viewed as a source node of a specific substar graph. Additionally, the node performs Rule 2 if it is also viewed as a relay node of another substar graph. These two rules are recursively performed for each other. Let $r_i(a)$ denote the last i symbols of label a. For example, $r_2(2314) = 14$. The two broadcasting rules are described as follows.

Rule 1. If a node u is the *source* node of the substar graph $S_i(r_{n-i}(u))$, for $2 \leqslant i \leqslant n$, the node u will send the message to its $i-1$ neighboring nodes $v_j = g_j(u)$, for $2 \leqslant j \leqslant i$. Each node v_j will be viewed as the *source* node of the substar graph $S_{j-1}(r_{n-j+1}(v_j))$ and it recursively applies the Rule 1 to broadcasting the message in the substar graph $S_{j-1}(r_{n-j+1}(v_j))$ for $2 \leqslant j \leqslant i$. Additionally, each of $i-2$ nodes v_j, for $2 \leqslant j \leqslant i-1$, is not only viewed as the *source* node of the substar graph $S_{j-1}(r_{n-j+1}(v_j))$ but also as the *relay* node of the substar graph $S_i(r_{n-i}(v_j))$. These nodes will also perform Rule 2.

For instance, a source node $u = 1234$ of the star graph S_4 is considered to broadcast the message to all of the other nodes. In the first time step, the source node 1234 sends the message to its three neighboring nodes $v_j = g_j(1234)$, for $2 \leqslant j \leqslant 4$, i.e., $g_2(1234) = 2134$, $g_3(1234) = 3214$, and $g_4(1234) = 4231$. The node $v_4 = 4231$ is the source node of the substar graph $S_3(1)$. The other two nodes $v_2 = 2134$ and $v_3 = 3214$ are not only viewed as the respective source nodes of the substar graphs $S_1(134)$ and $S_2(14)$ but also as the relay nodes of the substar graph S_4.

Rule 2. The *relay* node v_j of the substar graph $S_i(r_{n-i}(v_j))$ sends the message to $(i-j)$ neighboring nodes $w_k = g_k(v_j)$, for $j+1 \leqslant k \leqslant i$. Each node w_k will be viewed as the *source* node of the substar graph $S_{k-1}(r_{n-k+1}(w_k))$, for $j+1 \leqslant k \leqslant i$.

The above example is continued here. In the second time step, the source node 4231 of the substar graph $S_3(1)$ applies Rule 1 to sending the message to its neighboring nodes $g_i(4231)$, for $2 \leqslant i \leqslant 3$, i.e., $g_2(4231) = 2431$ and $g_3(4231) = 3241$. The node $v_2 = 2134$ can be viewed as the source node of the substar graph $S_1(134)$ and the relay node of the substar graph S_4. Therefore, the node v_2 will apply Rule 2 to sending the message to the neighboring nodes $w_k = g_k(2134)$, for $3 \leqslant k \leqslant 4$, i.e., $g_3(2134) = 3124$ and $g_4(2134) = 4132$. The nodes $w_3 = 3124$ and $w_4 = 4132$ are the respective source nodes of the substar graphs $S_2(24)$ and $S_3(2)$. Similarly, the node $v_3 = 3214$, a source node of the substar graph $S_2(14)$, applies Rule 1 to sending the message to the node $w_2 = g_2(3214) = 2314$, the source node of the substar graph $S_1(314)$. However, the node $v_3 = 3214$ which is also a relay node of the substar graph S_4 can apply Rule 2 to sending the message to the node $w_4 = g_4(3214) = 4213$, the source node of the substar graph $S_3(3)$. After 5 time steps, each node in the star graph S_4 can finally receive the message exactly once by tracing the above two rules as shown in Fig. 2. In addition, the tree paths of the broadcasting in S_4 are shown in Fig. 3.

A formal description of the broadcasting algorithm is given below. There are five arguments in the broadcasting algorithm which are denoted as the *message* to be broadcast, the label s of a node itself, the *dim*1 indicating the dimension of the substar graph with the source node s, the *flag* indicating whether the node s is regarded as either a source node of a substar graph or not only a source node but also a relay node of another substar graph, and the *dim*2 indicating the dimension of the substar graph in which the receiver is located.

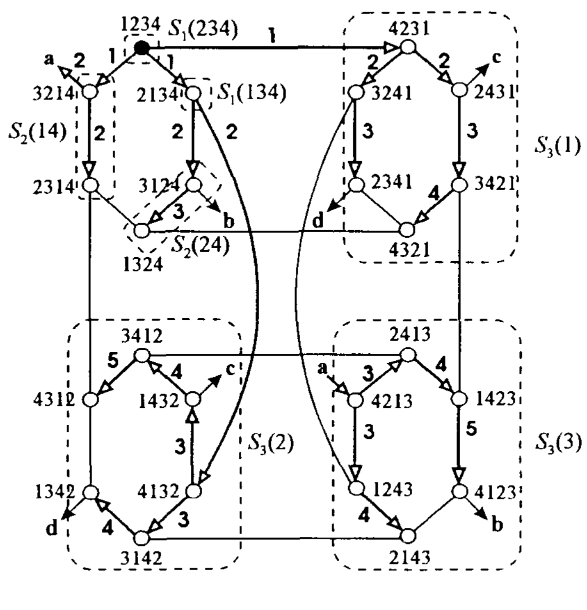

Fig. 2. Broadcasting in the star graph S_4.

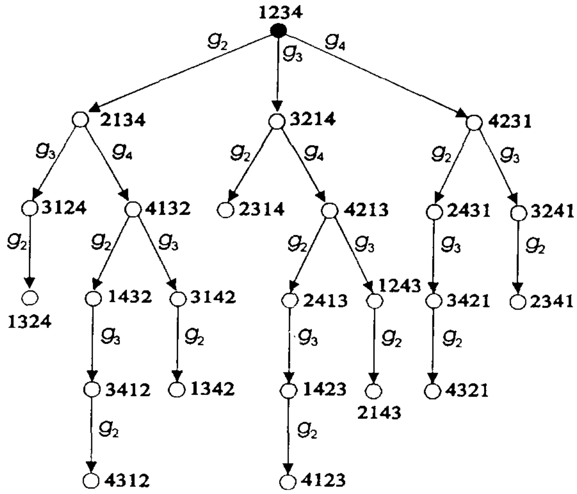

Fig. 3. The tree paths of the broadcasting in S_4.

Distributed Broadcasting Algorithm

Initialization: The arguments of the broadcasting procedure of the source node are:

$s \leftarrow$ the label of the source node, *dim*1 $\leftarrow$ the dimension of the star graph, *flag* $\leftarrow$ **true**, and *dim*2 $\leftarrow$ the dimension of the star graph.

procedure broadcasting (*message*, *s*, *dim*1, *flag*, *dim*2)
begin
  if *dim*1 = 1 **and** *flag* = **true then**
    stop; /* The node s is a source node of a one-dimensional substar graph. */
  if *dim*1 > 1 **then** /* Apply the Rule 1. */
    send (*message*, $g_{dim1}(s)$, *dim*1 − 1, true, *dim*1 − 1);
    /* Send the message to $g_{dim1}(s)$ which is viewed as a source node of the substar graph S_{dim1-1}. */
  for i = 2 **to** *dim*1 − 1 **do in parallel** /* Apply Rule 1. */
    send (*message*, $g_i(s)$, $i - 1$, **false**, *dim*1);
    /* Send the message to $g_i(s)$ which is viewed as not only a source node but also a relay node. */

```
if flag = false then /* Apply Rule 2. */
   for i = dim1 + 2 to dim2 do in parallel
      send (message, g_i(s), i − 1, true, i − 1);
      /* Send the message to the source node
      g_i(s) of the substar graph S_{i−1}. */
   stop;
end.
```

Theorem 4. *In an n-dimensional star graph S_n, the proposed broadcasting algorithm could complete in $2n-3$ time steps when $n \geqslant 2$.*

Proof. It shall be proven by mathematical induction.

Basic $n = 2$: It is trivial that the broadcasting algorithm takes one time step in the star graph S_2.

Induction hypothesis $n = m - 1$: The proposed broadcasting algorithm could complete in $2(m-1)-3$ time steps in the star graph S_{m-1}.

Induction $n = m$: By Lemma 3, it takes two time steps for the message of the S_m to be sent to one node of the other $m-1$ substar graphs S_{m-1}. When the source node of each substar graph S_{m-1} receives the message, these substar graphs can perform the broadcasting in parallel. By induction hypothesis, each substar graph S_{m-1} takes $2(m-1)-3$ time steps. The star graph S_m therefore takes $2(m-1)-3+2 = 2m-3$ time steps. □

Any optimal broadcasting algorithm on a graph with N nodes must take the time complexity $\Omega(\log N)$ when each node can send a message through at most one outgoing link at a time [1]. Any optimal broadcasting algorithm on a graph with the diameter d must, however, take the time complexity $\Omega(d)$ when each node can send a message through all of its outgoing links simultaneously. An n-dimensional star graph consists of $n!$ nodes and its diameter is $\lfloor 3(n-1)/2 \rfloor$ [2]. Any optimal broadcasting algorithm must therefore take the time complexity $\Omega(n \log n)$ in n-dimensional star graphs by sending a message through at most one outgoing link at a time. It can, however, only take the time complexity $\Omega(n)$ by sending a message through all of its outgoing links simultaneously. The proposed broadcasting algorithm here in a star graph S_n takes $2n-3$ time steps by sending a message through all of its outgoing links simultaneously.

3. Conclusions

In this paper, we have proposed a distributed broadcasting algorithm without sending message redundantly in the star graphs. The proposed algorithm merely takes $2n-3$ time steps in a star graph S_n. Since the message is received exactly once by each node, a stream of m messages can be broadcast within $O(m+n)$ in a pipeline fashion. In a real word, the assumption of simultaneous transmission is not reasonable for large n. However, the algorithm can be also used to send a message through a outgoing link at a time. Then the time complexity of broadcasting m messages is $O(mn + n^2)$.

References

[1] S.B. Akers, D. Harel and B. Krishnamurthy, The star graph: An attractive alternative to the n-cube, in: *Proc. 16th Internat. Conf. on Parallel Processing*, Pennsylvania, (1987) 393–400.

[2] S.B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, *IEEE Trans. Comput.* **38** (4) (1989) 555–565.

[3] S.G. Akl, K. Qiu and I. Stojmenovic, Data communication and computational geometry on the star and pancake interconnection networks, in: *Proc. Third IEEE Symp. on Parallel and Distributed Processing* (1991) 415–422.

[4] V.E. Mendia and D. Sarkar, Optimal broadcasting on the star graph, *IEEE Trans. Parallel Distributed Systems* **3** (4) (1992).