Broadcasting on Incomplete Star Graph Interconnection Networks

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Abstract

In this paper, we propose two one-to-all optimal broadcasting algorithms in incomplete star graphs. An incomplete star graph with N nodes, where $(n-1)! < N < n!$, is a subgraph of an n-star. Using a routing scheme to transmit a message to each substar composed of the incomplete star, our proposed broadcasting algorithm is optimal in $O(n \log n)$ on the single-port communication model. While broadcasting m messages on the incomplete star, we also present an optimal algorithm in $O(n \log n + m)$. Multi-message broadcasting is done first by transmitting m messages to each substar in a pipelined fashion and then by using the algorithm in [12] to broadcast them.

1 Introduction

Broadcasting, an important communication mechanism, is frequently used in many applications including areas of parallel algorithms, scientific parallel computing, and so forth, for message-passing multicomputers. Much research paid attention to the problems of broadcasting and personalized communication on a variety of interconnection networks [3] [4] [6] [9] [11] [13] [14] for achieving high-performance computing. Recently, an attractive interconnection network, star graph, to the hypercube topology has been proposed in [1] and [2]. More topological properties of the star graph can be found in [1]. Several broadcasting algorithms in star graphs have been presented by [9] [10] [11] [13] [14] [17]. The methods of constructing spanning trees in star graphs have also been addressed for solving the problems of personalized communication [4] [5] [3].

Although the interconnection networks including hypercube, star graph, and WK-recursive network have many nice properties, there is a serious restriction on the number of nodes within them. To relieve this restriction and eliminate the gap between the two consecutive sizes of a given topology, a variety of incomplete interconnection networks have been proposed with any number of nodes, such as the incomplete hypercubes [7], the incomplete stars [8], and incomplete WK-recursive networks [15]. Katseff proposed the broadcasting algorithm for the incomplete hypercubes [7]. Su, Chen, and Duh devoted to the broadcasting on the topology of incomplete WK-recursive networks [16]. For the incomplete star, the previous work [8] has designed the broadcasting algorithm on the special class called C^{n-1} defined latter. Thus, the purpose of this paper is to address broadcasting algorithms for the general incomplete star graphs.

In this paper, we propose two one-to-all optimal broadcasting algorithms for incomplete star graphs on the single-port communication model. We assume that two nodes via a link can communicate with each other simultaneously. An incomplete star graph with N nodes, where $(n-1)! < N < n!$, is a subgraph of an n-star. By partitioning the incomplete star into several substars to deliver the message, an optimal broadcasting in $O(n \log n)$ is addressed. Through this broadcasting scheme, we can generalize the previous work focused on a special class of incomplete stars. While broadcasting m messages on the incomplete star, we also present an optimal algorithm in $O(n \log n + m)$ based on the broadcasting scheme [12]. Multi-message broadcasting is done first by transmitting m messages to each substar in a pipelined fashion and then by using the algorithm proposed in [12] to broadcast them.

The rest organization of this paper is stated as follows. Section 2 introduces some terms, definitions, and lemmas used in this paper. In Section 3, how

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to perform one-to-all broadcasting with single-message and multi-message on the incomplete star is described. The two broadcasting algorithms are optimal in time. Finally, conclusions are summarized in Section 4.

2 Preliminaries

In this section, we introduce some notations and terms for star graphs and incomplete star graphs. A permutation of n distinct symbols in the set $\{1, 2, \ldots, n\}$ is represented by $(p_1, p_2, \ldots, p_n)$ where $p_i, p_j \in \{1, 2, \ldots, n\}$ and $p_i \neq p_j$ for $i \neq j$ and $1 \leq i, j \leq n$. A star graph with dimension n is an undirected graph in which the nodes with addresses correspond to the elements of the permutations of $\{1, 2, \ldots, n\}$ and the edges correspond to the actions of generators [1] [2]. The *generator* g_i is defined as the function $g_i(p_1 p_2 \cdots p_{i-1} p_i p_{i+1} \cdots p_n) = p_i p_2 \cdots p_{i-1} p_1 p_{i+1} \cdots p_n$ that interchanges p_i with p_1 for $2 \leq i \leq n$.

An undirected *star graph* with dimension n is denoted by $S_n = (P_n, E_n)$ where the set of vertices P_n is defined as $\{p_1 p_2 \cdots p_n \mid p_i, p_j \in \{1, 2, \ldots, n\}, p_i \neq p_j$ for $i \neq j$, $1 \leq i, j \leq n\}$ and the set of edges E_n is defined as $\{(v_1, v_2) \mid v_1, v_2 \in P_n, v_1 \neq v_2$, such that $v_1 = g_i(v_2)$ for $2 \leq i \leq n\}$. We use the notation S_n or n-star to denote an n-dimensional star graph in this paper.

Let $(p_1, p_2, \ldots, p_{n-i})$ be a permutation of $n-i$ distinct symbols in $\{1, 2, \ldots, n\}$ for $1 \leq i \leq n$. The *substar graph*, denoted by $S_n^i(p_1^{a_1}, p_2^{a_2}, \ldots, p_{n-i}^{a_{n-i}})$, is defined as a subgraph (V, E) of S_n where V is the set of nodes with the same $n-i$ symbols p_1 in position a_1, p_2 in position a_2, ..., p_{n-i} in position a_{n-i}, and E is the set of edges incident with any two of those nodes in V_{S_i}. Occasionally, we may denote $S_n^i(p_1^{a_1}, p_2^{a_2}, \ldots, p_{n-i}^{a_{n-i}})$ by a sequence $x_1 x_2 \cdots x_n$ such that for all $1 \leq i \leq k$ symbol $x_{p_i} = s_i$ and for all $j \notin \{p_1, p_2, \ldots, p_k\}$ symbol $x_j = *$, where a $*$ means a "don't care." For instance, $S_4^2(3^2, 1^4)$ is denoted as $*3*1$.

We define the incomplete star graph as follows [8]. An important aspect of the incomplete star is the methodology for constructing such a graph for an arbitrary number of nodes. Assume that N is the total number of nodes in the incomplete star where $(n-1)! < N < n!$ for some integer n. In this construction, we define the *coefficient vector* as an $(n-1)$-tuple $(b_{n-1}, b_{n-2}, \cdots, b_2, b_1)$, where $0 \leq b_i \leq i$, such that:

$$N = b_{n-1}(n-1)! + b_{n-2}(n-2)! + \cdots + b_2(2!) + b_1(1!) = \sum_{i=1}^{n-1} b_i(i!)$$

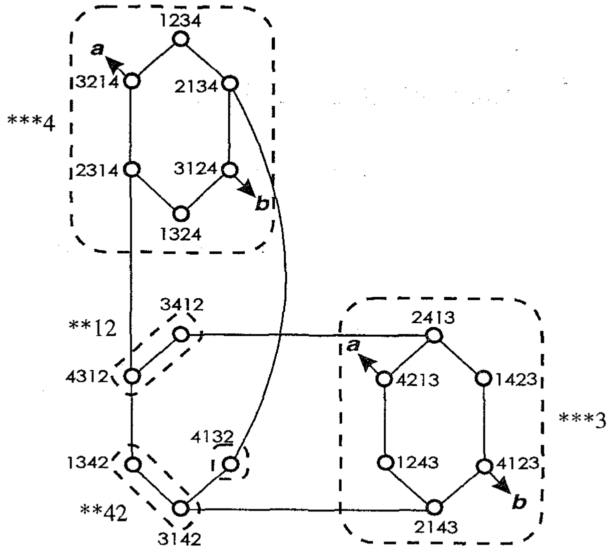

Figure 1: The incomplete star graph with 17 nodes.

That is, there exist b_{n-1} $(n-1)$-stars, b_{n-2} $(n-2)$-stars, ..., b_1 1-star in an N-node incomplete star graph.

The labeling scheme for each node in the incomplete star graph has been described in [8]. Those stars with the same dimension k are classified to form a class C^k. According to such a labeling scheme, two consecutive labels (stars) are said to be *adjacent*. Note that two adjacent substars need not belong to the same class. For example, a 17-node incomplete star graph with the coefficient vector $(b_3, b_2, b_1) = (2, 2, 1)$ consists of $C^3 = \{***4, ***3\}$, $C^2 = \{**12, **42\}$, and $C^1 = \{4132\}$ as shown in Fig. 1. The two substars $**12$ and $**42$ are adjacent in the same class C^2. The two substars $**42$ and 4132 are adjacent in different classes. For simplicity, we use the notation $*^k c_{k+1} \cdots c_n$ to denote the k-dimensional substar $** \cdots * c_{k+1} \cdots c_n$. Without loss of generality, the function *next*() as in [8] denotes the modulo operation for the integers unused in a lexicographical order. As the above illustration for the two adjacent substars $**12$ and $**42$, the value of *next*(1) is 4, i.e., $**next(1)2 = **42$.

In class C^k, we can number those substars to $C_1^k, C_2^k, \ldots$, and $C_{b_k}^k$ according to the labeling scheme. We call C_i^k the i-th substar in C^k. Hence, the two substars C_i^k and C_{i+1}^k for $1 \leq i < b_k$ are adjacent. We claim that each pair of substars in the same class has the adjacent property. That is, any node in substar C_i^k via constant links can connect to one of nodes in substar C_j^k, for $i \neq j$.

Lemma 1:

Suppose that there exist two substars C_i^k and C_j^k in the same class C^k where $i \neq j$, $1 \leq i, j \leq b_k$. Routing is done in at most two steps from any node in C_i^k to one of nodes in C_j^k.

Proof: Assume that $C_i^k = *^k ac_{k+2} \cdots c_n$ and $C_j^k = *^k bc_{k+2} \cdots c_n$. Suppose that there exists a node $c = x_1 x_2 \cdots x_k ac_{k+2} \cdots c_n$ in C_i^k where $x_s = b$, $1 \leq s \leq k$. The first step is to route the message from node c to node $d = g_s(c) = bx_2 \cdots x_{s-1} x_1 x_{s+1} \cdots x_k ac_{k+2} \cdots c_n$ in C_i^k. Then the second step routing is from node d to node $e = g_{k+1}(d) = ax_2 \cdots x_{s-1} x_1 x_{s+1} \cdots x_k bc_{k+2} \cdots c_n$. Clearly, the node e belongs to the substar C_j^k. Thus, this routing is done in at most two routing steps.

□

Corollary 1:

While sending message simultaneously on any two nodes on a pair of substars in C^k respectively, no conflict can be occurred.

Proof: For any first step routing in each substar in C^k, no conflict can be occurred. Then the second step routing is to apply the generator g_{k+1} to send message. Because all of the second routing steps are not to use the same links in C^k for any k, $1 \leq k \leq n-1$, the proof of this corollary is held.

□

In the following, we also claim that two substars according to the labeling scheme in the two consecutive classes also have the adjacent property. That is, any node in the last substar $C_{b_i}^i$ can connect to one of nodes in the first substar C_1^j, $i < j$, via $O(i-j)$ links [8].

Lemma 2:

Suppose that there exist two contiguous classes C^i and C^j for $i < j$. Routing is done in at most $2(i-j)+2$ steps from any node in the last substar $C_{b_i}^i$ to one of nodes in the first substar C_1^j according to the labeling scheme.

Proof: Assume that a node s is of the form

$$s = *^i a_{i+1} a_{i+2} \cdots a_n$$

in $C_{b_i}^i$ and a node d is of the form

$$d = *^j x_{j+1} x_{j+2} \cdots x_i next(a_{i+1}) a_{i+2} \cdots a_n$$

in C_1^j. Routing from s to s' as the following form

$$s' = *^j x_{j+1} x_{j+2} \cdots x_i a_{i+1} a_{i+2} \cdots a_n$$

in C_1^j is at most $2(i-j)$ steps because each symbol x_k, $j+1 \leq k \leq i$, located at its appropriate position needs to take at most 2 steps. Then, the node s' connects to the node

$$d = *^j x_{j+1} x_{j+2} \cdots x_i next(a_{i+1}) a_{i+2} \cdots a_n$$

in C_1^j via at most 2 links. Therefore, the total routing steps is at most $2(i-j)+2$.

□

Based on these terms and lemmas, we will address two broadcasting algorithms in the following section.

3 Broadcasting

In this section, we will introduce two one-to-all broadcasting algorithms on incomplete star graphs. The first one is for broadcasting a single message and the second is for broadcasting a stream of messages on the incomplete stars.

3.1 Single-message Broadcasting

First, the basic idea is described as follows. Assume that the source node in class C^i needs to deliver a message to all other nodes. First, the source node sends the message to one node of the first substar in C^i according to the labeling scheme. This node delivers the message to one node of the last substar in C^i and to one node of the last substar in the previous class simultaneously. To recursively deliver the source message in this manner, one node of each substar in all of classes can receive the message from the source node. We know that a class is composed of substars with the same dimension. For each class, the node with source message is to transmit the message to other substars. Thus, one node of each substar in the incomplete star has received the message. By the broadcasting algorithm in [9] [11] [14], we can independently perform the one-to-all broadcasting on each substar in parallel.

In the following lemma, we shall prove how many steps are required to complete the processing that one node of the first substar in each class received the message from the source node.

Lemma 3:

It takes at most $6n-10$ steps that the source node sends the message to all of one node of the first substar in their corresponding class.

Proof: We consider the worst case that there are several substars in each class and the source node is in

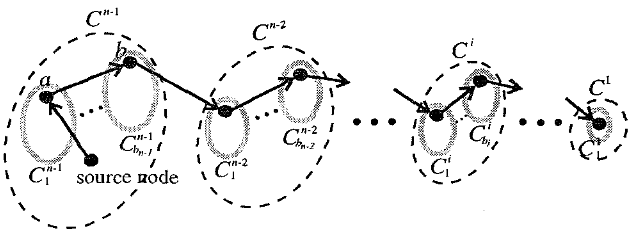

Figure 2: Routing among classes.

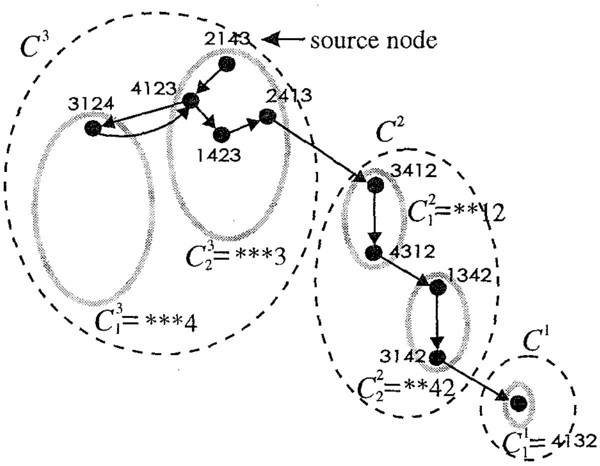

Figure 3: Routing on the 17-node incomplete star.

the class C^{n-1}. This is because that if the source node is located in the middle of these classes, the communication routing will be independently along the two opposite directions in parallel. This is not the worst case.

Thus, we assume that there exist substars in each class of C^{n-1}, C^{n-2}, ..., and C^1. The sketch proof can be easily examined from the Fig. 2. By Lemma 1, the node a sends the message to the one node b of the last substar, which is to take 2 steps. By Lemma 2, the node b sends the message to the node of the first substar in class C^{n-2}, which is to take $2((n-1)-(n-2))+2=4$ steps. While proceeding the two previous operations to route the message to the node in C^1_1, the total steps is:

$$2+\sum_{i=1}^{n-2}(2+4)=6n-10.$$

□

For example, we consider the broadcasting in a 17-node incomplete star as shown in Fig. 3. Assume that the source node is 2143 in ***3. First, the node 2143 sends the source message to node 3124 of the first substar C^3_1 in C^3 via node 4123. The node 3124 sends the message to 4123 of the last substar C^3_2 in C_3. The node 4123 sends the message to 2413 via 1423. The node 2413 sends message to 3412 of the first substar C^2_1 in C^2. The node 3412 sends the message to 1342 of the last substar C^2_2 in C^2 via 4312. The node 1342 delivers the message to 4132 in C^1_1 via 3142. Hence, the nodes of 3124, 3412, and 4132 on the first substars in classes C^3, C^2, and C^1 received the message from the source node, respectively.

The above mentioned process is to route message among classes. In the following, we describe the routing among substars in the same class. We shall derive how many steps are required to complete the processing that one node of each substar in class C^i received the message from the node of the first substar in class C^i as follows.

Lemma 4:

In class C^i, there are substars C^i_1, C^i_2, ..., and $C^i_{b_i}$. *Assume that a node a_1 in C^i_1 has received the broadcasting message. It takes at most $2\lceil\log b_i\rceil$ steps that node a_1 delivers the message to one node of other substars in C^i.*

Proof: One node a_1 in C^i_1 sends the message to one node a_2 in C^i_2, which is to take at most 2 steps. From Corollary 1, a_1 and a_2 in parallel send message to nodes a_3 and a_4 in C^i_3 and C^i_1, respectively. Finally, it takes at most $2\lceil\log b_i\rceil$ steps to broadcast the message to one node in class C^i by using the recursively doubling scheme.

□

For example, there are eight substars C^i_1, ⋯, C^i_8 in class C^i. First, one node with the message in C^i_1 delivers to one node in C^i_2. Then, the two nodes in C^i_1 and C^i_2 deliver the message in parallel to the two nodes in C^i_3 and C^i_4, respectively. Finally, the four nodes in C^i_1, C^i_2, C^i_3, and C^i_4 send the message to the nodes in C^i_5, C^i_6, C^i_7, and C^i_8, respectively, in parallel. The whole processing mentioned above can be shown in Fig. 4. The basic time step t is at most 2.

Based on the above processing, one node within all of substars composed of the incomplete star received the message from the source node. Next, we can perform broadcasting on each substar by the proposed algorithms in [9], [11], or [14] to accomplish this operation. For the problem of one-to-all broadcasting, the lower bound of time complexity is $O(\log N)=O(n\log n)$, where $(n-1)!<N<n!$.

Theorem 1:

The single-message broadcasting algorithm takes

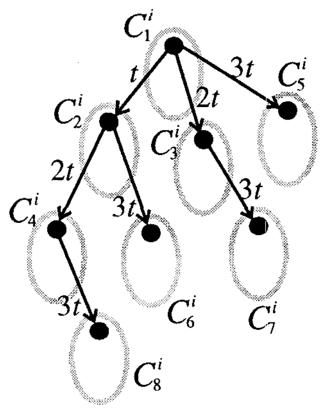

Figure 4: Routing in class C^i.

$O(n \log n)$, which is optimal, for an N-node incomplete star.

Proof: In Lemma 3, first, it takes $O(6n-10) = O(6n)$. In Lemma 4, this process takes at most $O(\max_{1 \le i \le n-1}(2\lceil \log b_i \rceil)) = O(\log(n-1))$. After performing Lemmas 3 and 4, one node of each substar composed of the N-node incomplete star has received the broadcasting message. By the proposed algorithms as in [9], [11], or [14], the broadcasting time in n-star is $O(n \log n)$. The largest size of substar in the incomplete star is $n-1$. Thus, the time complexity of our proposed algorithm is

$$O(6n)+O(\log(n-1))+O((n-1)\log(n-1)) = O(n \log n)$$

which is optimal.

□

3.2 Multi-message Broadcasting

Due to the need to transmit a large amount of data to all other nodes for some applications, we may divide the original data into several packets (messages) to do broadcasting. In this subsection, an optimal algorithm is presented for multi-message broadcasting on incomplete stars. The basic idea for broadcasting is described as follows.

Assume that the source node in class C^i needs to deliver a stream of m messages to all other nodes. First, we will build a path to connect each class. The source node can be connected with one node of the first substar in C^i. This node of the first substar can be connected to one node of the last substar in this class and to one node of the last substar in the previously consecutive class. We recursively construct a routing path so that the path connects the first substar and the last substar on each class among classes. Thus, we can transmit these messages in a pipelined fashion via this established path. By the same idea of this construction, we can build a path connecting each substar in class C^i. By this way, we can also transmit the messages in a pipelined fashion via this constructed path. So far, one node of each substar in the incomplete star has received the m messages. By the broadcasting algorithm in [12], we can independently perform the one-to-all broadcasting with multiple messages on each substar in parallel.

By Lemma 3, we can construct a path with length of at most $6n-10$, which connects the nodes of the first substar in each class. In this constructed path, we may use the same links but different directions. If the source node is in C^{n-1} or C^1, transmitting a stream of m messages in a pipelined fashion requires to take $O(n+m)$ time steps. Otherwise, it costs two times of $O(n+m)$ at most. Thus, this operation takes $O(n+m)$.

Within a class C^i, the transmission of m messages proceeded in a pipelined fashion costs $O(b_i+m)$ for $1 \le i \le n-1$. Due to the operation performed in parallel for each class, it takes $O(\max_{1 \le i \le n-1} b_i + m) = O(n+m)$ time steps that one node of each substar received m messages from the node of the first substar with these messages.

After executing the above two procedures, we can apply the proposed algorithm in [12] to accomplish this broadcasting. In [12], the time of broadcasting m messages to all other nodes for an n-star is $O(n \log n + m)$, which is time optimal. Thus, it is optimal for broadcasting m messages to all other nodes for an N-node incomplete star in $O(n \log n + m)$. Our proposed broadcasting algorithm takes $O(n \log n + m)$ which will be proven below.

Theorem 2:

The multi-message broadcasting algorithm is to take $O(n \log n + m)$, which is optimal, for an N-node incomplete star.

Proof: We know that the path we constructed is with length of at most $6n-10$ which connects the nodes of the first substar in each class. Transmitting a stream of m messages in a pipelined fashion requires to take $O(n+m)$ time. For each class, it requires to take $O(n+m)$ time that one node of each substar received m messages from the node of the first substar with these messages. One node of each complete substar will deliver these m messages to all other nodes, which takes $O(\max_{1 \le i \le n-1}(i \log i + m)) = O((n-1)\log(n-1)+m)$. Hence, the time complexity of this algorithm is

$$O(n+m) + O(n+m) + O((n-1)\log(n-1) + m)$$
$$= O(n \log n + m)$$

which is optimal.

□

4 Conclusions

In this paper, two optimal broadcasting algorithms were proposed on incomplete stars both based on the single-port communication model. The first one takes optimal time, $O(n \log n)$, for one-to-all broadcasting in an N-node incomplete star where $(n-1)! < N < n!$. The second optimal algorithm presented is in $O(n \log n + m)$ for broadcasting m messages in a pipelined fashion on the incomplete star.

Acknowledgements

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