Bubblesort Star Graphs: A New Interconnection Network

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Abstract

In this paper, we propose and analyze a new interconnection network called bubblesort star graph, which is the merger of the bubblesort graph and the star graph. We present the deadlock-free wormhole routing algorithm for the proposed network. We also develop the method to embed a mesh into a bubblesort star graph with dilation two and expansion one. Besides, we use the recursive scheme to embed the multiple disjoint copies of the hypercube into a bubblesort star graph with all faults recovery capacity as well as constant expansion and dilation one or two. This reflects the fact that the embeddability of the bubblesort star graph is much better than that of the star graph.

Keywords: *graph embedding, hypercube, star graph, wormhole routing.*

1 Introduction

Due to the rapid progress of VLSI and the success of commercial available multicomputers, there are many static message-passing interconnection networks proposed for the parallel algorithms design such as complete binary trees, rings, tori, and hypercubes; and further, there are many algorithms developed on these architectures. Specially, the algebra computation problems can be easily solved on the torus and recursive divide-and-conquer algorithms are suitable for the hypercube. Recently, there are many topologies proposed continually. If we also want to propose a new topology, what are the benchmarks for us to judge the "goodness" of an interconnection network? In [1, 2, 8, 10, 11], they suggest that cost metrics should include degree, diameter, average diameter, connectivity, and fault diameter. Besides, there are some more complex attributes needed to be considered: genus, optimal algorithms for various models of packet communication, symmetry properties, recursive scalability, embeddability, and so on.

The star graph [1, 2] proposed recently is hierarchical, vertex and edge symmetric, maximally fault tolerance, and strongly resilient. In addition, it has been shown that the star graph [1, 2, 8, 10, 11] possesses smaller degree, diameter, genus and fault diameter than those of the hypercube when these two topologies have almost the same number of vertices. Although the star graph is superior to the hypercube from some standpoints of the graph theory, yet the embeddability of the star graph is poor. We list some facts about the one-to-one embedding on the star graph as follows. First, there is no dilation one embedding of a hypercube with dimension k (denoted $H(k)$ or k-cube), for all $k > 1$, into a star graph with dimension n (abbreviated $S(n)$) for all n [16]. Secondly, there is no dilation one embedding of a k-dimensional mesh into a star graph $S(n)$ for all $k > 1$ and for all n [9, 17]. Thirdly, there is no dilation one embedding of n-level complete binary tree into a star graph $S(n)$ for all n [12]. Furthermore, let us consider the dilation two embedding of topologies into the star graph. In [16], M. Nigam et al. said that there is a dilation two embedding of an $H(n)$ into a star graph $S(2^n)$ for all n. However, the expansion of this embedding $((2^n-1)!)$ is too intolerably large to accept. Additionally, J.S. Jwo et al. indicated that two-dimensional mesh of size $2 \times (n!/2)$ can be embedded into an $S(n)$ with unit expansion and dilation two [9]. Unfortunately, this kind of mesh is rarely used because of its low bisection width (2) and long diameter $(n!/2)$.

In many literatures [6, 9, 13, 16, 17], we know that embeddability is important in that at least if a network $\mathcal{A}$ can be embedded in a network $\mathcal{B}$ with small dilation and expansion, then all the algorithms developed on $\mathcal{A}$ can be transported onto $\mathcal{B}$ efficiently and effectively. Motive of this reason, we would like to propose a new variant of the star graph. In this paper, we will use the merging approach

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to propose the new interconnection network—bubblesort star graph, which is the *merger* [7] of the bubblesort graph [2] and the star graph. (Originally, the merger graph is used to achieve the edge fault tolerance [7].) Clearly, the star graph owns many attractive properties except the embeddability as well as the bubblesort graph is simple and possesses some desirable features except the long diameter. So we may expect that the bubblesort star graph will combine the advantages of both graphs and surmounts their individual flaws.

Later, we can know that the degree, diameter, and genus of the bubblesort star graph is smaller than those of the hypercube when these two graphs have approximately the same number of vertices. This implies that bubblesort star graph should have a more efficient layout than the hypercube. Above all, it seems to be worthwhile to increase about twice cost of the degree of the star graph (which turns into the *bubblesort star graph*) in order to improve some performance such as smaller average distance, higher bisection width, lower traffic density, shorter broadcasting time, stronger network resilience. Moreover, we can embed a mesh into a bubblesort star graph with dilation two and unit expansion. We can also embed the multiple disjoint copies of the hypercube into a star graph with all fault recovery capacity as well as constant expansion and dilation one or two. This reflects the fact that the embeddability of the bubblesort star graph is much better than that of the star graph.

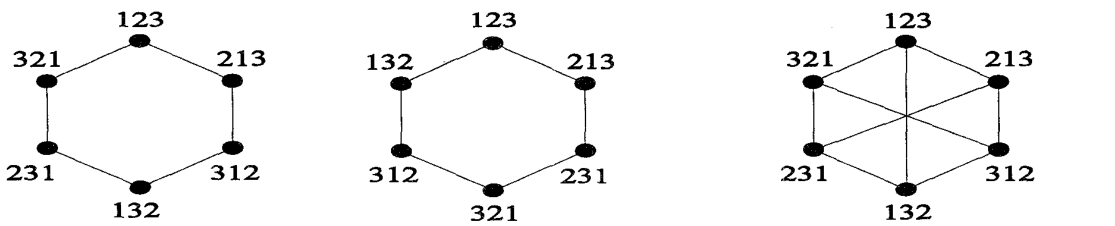
The remainder of this paper is organized as follows. In Section 2, we present the topological properties of the $BS(n)$ and wormhole routing algorithm for both of the $S(n)$ and $BS(n)$. In Section 3, we analyze and present the methods for embedding the mesh and multiple disjoint copies of the hypercube into the bubblesort star graph with dilation one or two as well as all faults recovery capacity. Lastly, the conclusions are drawn in Section 4. Note that some of the theorems in this paper will be omitted due to the limitation of space. Details and some related works on the bubblesort star graph including some topological properties, data communications, fault-tolerant capacities, terminal/network reliabilities, and graph layout properties can be found in [4].

2 Preliminaries

Interconnection networks used to pass messages containing data and control information can be modeled as a graph where the vertices correspond to the processors and the edges correspond to the communication channels. For a graph $G = \langle V_G, E_G \rangle$, let V_G and E_G denote its vertex set and edge set, respectively. The cardinalities of the vertex set and edge set are denoted by $|V_G|$ and $|E_G|$ respectively. For $u, v \in V_G$, the degree of the vertex u is denoted by $deg(u)$ and the distance between u and v is denoted by $dist(u,v)$. Let $\mathcal{S}_n$ be an ordered set of n distinct symbols and $\mathcal{S}_n = \{1, 2, 3, \ldots, n\}$. Let $Perm(\mathcal{S}_n)$ denote the set of permutations over the set $\mathcal{S}_n$. Clearly, the permutation set $Perm(\mathcal{S}_n)$ together with the *composition* operation "$\circ$" forms the permutation group [1, 2, 10]. In general, any permutation $p \in Perm(\mathcal{S}_n)$ can be expressed as the product of k disjoint cycles and l invariants: $p = p_1 p_2 \cdots p_n = C_1\, C_2 \cdots C_k\, e_1\, e_2 \cdots e_l$ where $C_i = (j_1\, j_2 \cdots j_m)$, $2 \le i \le k$, $m \ge 2$, and $j_\ell \in \mathcal{S}_n$ are distinct, and $p_{e_j} = e_j$ [10]. For convenience, we omit to write the invariants when p is represented by its cycle structure. If cycles with the form $(i_1\, i_2 \cdots i_k)$ such that $i_{j+1} = i_j + 1$ or $i_{j+1} = i_j - 1$, for all $1 \le j \le k-1$ and $k \ge 2$, are called *straight cycles*. For example, let $\mathcal{S}_3 = \{1, 2, 3\}$, then $Perm(\mathcal{S}_3) = \{123, 132, 213, 231, 312, 321\}$; besides, if $p = 132$, then p can be expressed as $p = (23)$ and $p \circ (1,3) = 132 \circ (1,3) = 231$.

Then the *star graph* $S(n) = \langle V_{S(n)}, E_{S(n)} \rangle$ which we are of interested is defined as $V_{S(n)} = Perm(\mathcal{S}_n)$ and $E_{S(n)} = \{(u,v) : u, v \in Perm(\mathcal{S}_n)$ and $v = u \circ (1,i)$ for some $2 \le i \le n\}$. The *bubblesort graph* $B(n) = \langle V_{B(n)}, E_{B(n)} \rangle$ is defined as $V_{B(n)} = Perm(\mathcal{S}_n)$ and $E_{B(n)} = \{(u,v) : u, v \in Perm(\mathcal{S}_n)$ and $v = u \circ (i-1, i)$ for some $2 \le i \le n\}$. That is, $u, v \in Perm(\mathcal{S}_n)$ and u, v are neighbors in $V_{B(n)}$ if there exists $u = v \circ (i-1, i)$ for some i, where $2 \le i \le n$. We define $BS(n) = \langle V_{BS(n)}, E_{BS(n)} \rangle$ to be the ***bubblesort star graph*** with the vertex set $V_{BS(n)} = Perm(\mathcal{S}_n)$ and edge set $E_{BS(n)} = \{(u,v) : u, v \in Perm(\mathcal{S}_n)$ and $(v = u \circ (1,i),\ 2 \le i \le n)$ or $(v = u \circ (i-1, i),\ 3 \le i \le n)\}$. Figure 1(a), (b), and (c) show the samples of the star graph, bubblesort graph, and the bubblesort star graph with the dimension 3 respectively. From the definition of the $BS(n)$, we can know that $BS(n)$ is the merger of the $B(n)$ and $S(n)$ [7]. It is easy to see that $|V_{BS(n)}| = n!$ and $|E_{BS(n)}| = \frac{n! \times (2n-3)}{2}$ for all $n \ge 2$. Besides, for all $u \in V_{BS(n)}$, $deg(u) = 2n-3$. Since the bubblesort star graph is a Cayley graph, it is indeed regular and vertex symmetry; moreover, the bubblesort star graph is Hamiltonian and bipartite. In the rest of this section, we will introduce some topological properties of the bubblesort star graph.

First, the diameter of the $BS(n)$ is $\lfloor \frac{3(n-1)}{2} \rfloor$ for all $n \ge 4$ since the bubblesort star graph inherits some topological properties of the star graph. Although the diameter of the $BS(n)$ is equal to that of the $S(n)$, we will find that $BS(n)$ has many advantages over the $S(n)$. At least, now, we can see that the average diameter (or average distance) of the $BS(n)$ is shorter than that of the $S(n)$. Hereby we would like to present the routing algorithm in order to derive the upper bound of the average distance of the $BS(n)$. Because the bubblesort star graph is vertex symmetry, this implies that routing between two vertices is equivalent to sorting a permutation [2]. As indicated in [2, 3, 5], routing function should be as simple as possible in order to reduce the hardware complexity and processing delay. Owing to this reason, we present the heuristic routing. Let $p = C_1\, C_2 \cdots C_k \in V_{BS(n)}$. In principle, we can follow the routing algorithm in the star graph to route the message from the node p to the identity node $Id = 123 \cdots n$. Once there exists a straight cycle $C_j = (j_1, j_2, \ldots, j_i)$ of length i, for $1 \le j \le k$ and $2 \le i \le n-1$, in



(a)star graph S(3) **(b)bubblesort graph B(3)** **(c)bubblesort star graph BS(3)**

Figure 1: (a), (b), and (c) show the samples of the star graph, bubblesort graph, and the bubblesort star graph with the the dimension 3 respectively.

p without containing 1, then we can route a message from p to the intermediate node (perhaps includes the Id) by applying the generators (i_k, i_{k-1}), (i_{k-1}, i_{k-2}),..., $(2,1)$ successively, which is through $i-1$ hops and fewer than that in the $S(n)$ by 2. Let us route a permutation $p = 1745632 = (2\,7)(3\,4\,5\,6) \in V_{BS(7)}$, as the source node, to the $Id = 1234567$ as an example. Let $p \xrightarrow{(i,j)} q$ represent that node p sends the message to node q via the generator (i,j). $p = 1745632 \xrightarrow{(5,6)} 1745362 \xrightarrow{(4,5)} 1743562 \xrightarrow{(3,4)} 1734562 \xrightarrow{(1,2)} 7134562 \xrightarrow{(1,7)} 2134567 \xrightarrow{(1,2)} 1234567$. According to the above description, we have the upper bound of the average distance of the $BS(n)$. Given a graph G, let $\overline{d}_G$ denote the average distance of the graph G.

Theorem 2.1 $\overline{d}_{BS(n)} \leq \overline{d}_{S(n)} - \frac{2}{n-1} + \frac{4}{n!}$, where $\overline{d}_{S(n)} = n - 4 + \frac{2}{n} + \sum_{i=1}^{n} \frac{1}{i}$.

Proof: Let $p = C_1\, C_2 \cdots C_k \in V_{BS(n)}$. We know that if there exists a straight cycle C_j of length i, for all $2 \leq i \leq n-1$, in p without containing 1, then we can route a message from p to the intermediate node (perhaps includes the Id) through $i-1$ hops which is fewer than that in $S(n)$ by 2. The number of this kind of straight cycles of length i in all permutations is $2(n-i)\times(n-i)!$ for all $3 \leq i \leq n-1$. When $i = 2$, there are $(n-2)\times(n-2)!$ straight cycles of length 2 in all permutations. Hence,

$$\overline{d}_{BS(n)} \leq \overline{d}_{S(n)} - \frac{2}{n!}\left\{(n-2)(n-2)! + 2\sum_{i=1}^{n-3} i \times i!\right\}.$$

As we know that $n! = 0! + \sum_{i=1}^{n-1} i \times i!$, the theorem thus follows. □

Now, we will use the negative-hop scheme [3] to present the distributed wormhole routing algorithm which is suitable for both of the star graph $S(n)$ and bubblesort star graph $BS(n)$; besides, the number of virtual channels required ($\lfloor \frac{3n}{4} \rfloor$) are less than that $(n-1)$ proposed by J.Mišić and Z.Jovanović [15]. In the *wormhole* model, each packet consists of a sequence of elementary flow control digits (*flits*). During each step, each flit of a packet can advance across a channel in a pipelined fashion. An adaptive wormhole routing algorithm can be defined in two parts, one is the routing relation and the other is the selection function. The routing relation (function) specifies a set of virtual channels that may be used for the next step based on the current channel, destination address, and some routing information. The set of virtual channels determined will be called the adaptive set (AS). The selection function is used to pick the next channel from the AS to route according to the network congestion information.

Let $dia(G)$ denote the diameter of a graph G. If G is bipartite, then we merely need $1+\lfloor dia(G)/2 \rfloor$ virtual channels on each physical link of G when we take the negative-hop scheme [3]. Initially, the message is routed in the zeroth virtual channel. During the routing, the message transfer to the virtual channel one class higher than what it reserved in the previous hop only when it moves from an odd permutation to an even permutation. Clearly, we can find that the classes of virtual channels where the message routes result in an increasing order, this routing method is thus deadlock-free.

/⋆ p is the current node address and destination is the identity permutation Id; $p = p_1\, p_2 \cdots p_n$ can be expressed as a product of disjoint cycles $p = C_1\, C_2 \cdots C_k$ ⋆/
Algorithm ADAPTIVE_ROUTE;
/⋆ Current node p receives a header flit (*msg, next, vc, NH*) ⋆/
/⋆ Notations: *msg*: message, *next*: next node address, *vc*: class of virtual channels, *NH*: the number of Negative-Hops; If p is the source node, then *vc*=*NH*=0 ⋆/

begin
1 **if** $p = Id$ **then stop**
2 **else**
3 **begin**
4 $vc := NH$;
5 **if** Current node address is an odd permutation **then**
6 $NH := NH+1$;
7 $AS := \emptyset$;
/⋆ Initially, AS is assigned to the empty set. In addition, lines 9 ∼ 14 are used to determine the adaptive set AS. ⋆/
8 **if** $(\exists\ C_i = (1, i_1, i_2, \ldots, i_k))$ **then**
9 $AS := AS \cup \{p \circ (1, i_1)\}$;

```
10    while (∃ C_i = (i_1 i_2 ... i_k), C_i is not a straight
          cycle and i_j ≠ 1 ∀ j )
11        AS := AS ∪ { p ∘ (1, i_j): ∀ j };
12    while (∃ C_i = (i_1 i_2 ... i_k) and (i_{j+1} = i_j + 1
          or i_{j+1} = i_j − 1, ∀ 1 ≤ j ≤ k − 1))
13        AS := AS ∪ { p ∘ (i_{k−1}, i_k) };
14    randomly select the next node from AS;
15    send header flit (msg, next, vc, NH) to the next
          node along the virtual channel vc;
16  end
end
```

Here we give an example. Let $p \stackrel{(i,j);vc}{\longrightarrow} q$ denote that node p send flits to node q through the link(generator) (i, j) with the class of virtual channel vc. A message originating at source node $p = 164352 = (26)(34)$ which is an even permutation is to be routed to destination $Id = 123456$ in $BS(6)$. We have eight possible paths from $p = 164352$ to $Id = 123456$ according to the ADAPTIVE_ROUTE algorithm and we list only two of them:

(1) $164352 \stackrel{(3,4);0}{\longrightarrow} 163452 \stackrel{(1,2);0}{\longrightarrow} 613452 \stackrel{(1,6);1}{\longrightarrow} 213456 \stackrel{(1,2);1}{\longrightarrow} 123456$.

(2) $164352 \stackrel{(1,2);0}{\longrightarrow} 614352 \stackrel{(1,6);0}{\longrightarrow} 214356 \stackrel{(3,4);1}{\longrightarrow} 213456 \stackrel{(1,2);1}{\longrightarrow} 123456$.

Note that the possible paths provided by the ADAPTIVE_ROUTE algorithm are non-minimal. Besides, by the previous discussion, we know that we only need $1 + \lfloor \frac{1}{2} dia(BS(n)) \rfloor = 1 + \lfloor \frac{1}{2} \lfloor \frac{3}{2}(n-1) \rfloor \rfloor$ virtual channels for each physical link in the star graph or the bubblesort star graph. Thus if n is odd, we can use only $\lfloor \frac{3n+1}{4} \rfloor$ virtual channels for each physical channel and if n is even, we merely need to split each physical channel into $\lfloor \frac{3n}{4} \rfloor$ virtual channels. If line 13 and 14 are deleted from the ADAPTIVE_ROUTE algorithm, then this algorithm is suitable for the star graph and becomes the *minimal* fully adaptive routing algorithm.

3 Graph Embeddings

Let G and H be two undirected graphs that model two sets of interconnection networks. An *embedding* ϕ of G into H is a mapping of V_G into V_H and of E_G into simple paths of H. In this section, we only consider the one-to-one embedding $\phi : V_G \to V_H$. That is, if $u, v \in V_G$ and $u \neq v$, then $\phi(u) \neq \phi(v)$. The ratio $|V_H|/|V_G|$ is called the *expansion* of the embedding. Let u and v be any two adjacent nodes in G. The dilation of an edge $(u, v) \in E_G$ is $dist(\phi(u), \phi(v))$. The *dilation of the embedding* ϕ is $max\{ dist(\phi(u), \phi(v)) \}$ for all $(u, v) \in E_G$, and the average dilation of the embedding ϕ is

$$\frac{1}{|E_G|} \sum_{(u,v) \in E_G} \{ dist(\phi(u), \phi(v)) \}.$$

Given two graphs, G and H, if we can embed the graph G into the graph H with dilation d, then we write $G \sqsubseteq H^d$. (When $d = 1$, we may write $G \sqsubseteq H$.)

When we talk about the embedding, we cannot neglect the ability of the fault tolerance and multiple network embeddings. Since with the increasing size of multiprocessors, it maybe results in higher probability of faulty nodes. When a fault occurs, it may consume much of computation time to recover the fault if without a proper recovery scheme. In addition, A.K. Gupta and S.E. Hambrusch [6] indicate that in the MIMD environment, system will allow the multiple users to simulate a great number of networks simultaneously, thus the problem of how to efficiently embed multiple graphs will become critical. In this section, we will solve all of these problems.

3.1 Torus Embedding

Since the image processing and algebra computation problems can be easily solved on the torus or mesh and W.J. Dally et al. [5] argue that the low dimensional torus (or mesh) outperforms the n-cube under the constant wire bisection, it is worthwhile to discuss the 2-dimensional torus embedding.

Let $T(m \times n)$ represent the 2-dimensional *torus* of size $m \times n$ and $N(i, j)$ be the i-th row and j-th column node, where $0 \le i \le m-1$ and $0 \le j \le n-1$. Two nodes $N(i, j)$ and $N(i', j')$ in the torus $T(m \times n)$ are neighbors, if and only if, either $i' = i \pm 1 \pmod m$ and $j' = j$ or $i' = i$ and $j' = j \pm 1 \pmod n$.

Let $\mathcal{S}_n = \{1, 2, \ldots, n\}$ and π_i^u represent the symbol of i-th position in u, where $u \in Perm(\mathcal{S}_n)$. Given two arbitrary integer strings u and v, $u \| v$ denotes the concatenation of u and v. If $u, v \in Perm(\mathcal{S}_n)$ and we say that $u = v$, if and only if, $\pi_i^u = \pi_i^v$ for all $1 \le i \le n$.

Lemma 3.1 [4] Let $u, v \in Perm(\mathcal{S}_{n-1})$, and n be a symbol such that $n \notin \mathcal{S}_{n-1}$. Let $k_1, k_2 \in \mathcal{N}$, $1 \le k_1, k_2 \le n$, and $u' = (n \| u) \circ (1, k_1)$, $v' = (n \| v) \circ (1, k_2)$. Thus, if $u \neq v$ or $k_1 \neq k_2$, then $u' \neq v'$.

Theorem 3.2 The 2-dimensional torus of size $n \times (n-1)!$ can be embedded into the bubblesort star graph $BS(n)$ with dilation two and expansion one. Besides, the average dilation of the embedding is $\frac{3}{2} - \frac{1}{n}$.

Proof. Let $\mathcal{S}_{n-1} = \{1, 2, \ldots, n-1\}$ and the vertex set of the bubblesort graph $B(n-1)$ be the set of permutations over $\mathcal{S}_{n-1}$. We know that the bubblesort graph $B(n-1)$ which is the subgraph of the $BS(n-1)$ is Hamiltonian. Let the vertices in the Hamiltonian cycle of the $B(n-1)$ be denoted by $U_0, U_1, U_2, \ldots, U_{p-1}$ in turn, where $p = (n-1)!$. It is clear that $(U_{k-1}, U_k) \in E_{B(n-1)}$ for all $1 \le k \le p-1$ and also $(U_{k-1}, U_k) \in E_{BS(n-1)}$. Let $BS(n) = \langle V_{BS(n)}, E_{BS(n)} \rangle$ and $V_{BS(n)} = Perm(\mathcal{S}_n)$, where $\mathcal{S}_n = \mathcal{S}_{n-1} \cup \{n\}$. Now, we define the embedding, ϕ, by mapping the node $N(i, j)$ in the $T(n \times (n-1)!)$, for $0 \le i \le n-1$ and $0 \le j \le p-1$, to the node $(n \| U_j) \circ (1, 1+i)$ in the $BS(n)$.

From Lemma 3.1, it follows that the embedding ϕ is one-to-one. The proof of the *unit expansion* of the embedding is thus reached. Therefore, we take account of the dilation.

From the definition of the torus networks, we know that the edge $(N(i,j),\ N(i',j'))$ belongs to the edge set $E_{T(n\times(n-1)!)}$, if and only if, either $i' = i \pm 1 \pmod n$ and $j' = j$ or $i' = i$ and $j' = j \pm 1 \pmod p$. Without loss of generality, we only consider two cases; one is $i' = i+1 \pmod n$ and $j' = j$, and the other is $i' = i$ and $j' = j+1 \pmod p$, where $0 \le i \le n-1$ and $0 \le j \le p-1$.

CASE 1: $i' = i+1 \pmod n$ and $j' = j$.

Node $N(i,j)$ in the torus is mapped to the $BS(n)$ node $(n\|U_j)\circ(1,1+i)$ and its neighbor $N(i',j)$ is mapped to the node $(n\|U_j)\circ(1,1+i')$. Thus, if $i = 0$ or $i = n-1$, then $\phi(N(i',j)) = \phi(N(i,j))\circ(i+1, i+2 \pmod n)$. If $1 \le i \le n-2$, then $\phi(N(i',j)) = \phi(N(i,j))\circ(i+1,i+2)\circ(1,i+1)$.

CASE 2: $i' = i$ and $j' = j+1 \pmod p$.

Node $N(i,j)$ in the torus is mapped to the $BS(n)$ node $(n\|U_j)\circ(1,1+i)$ and its neighbor $N(i,j')$ is mapped to the node $(n\|U_{j'})\circ(1,1+i)$. As mentioned above, we know that $(U_j, U_{j'})$ belongs to the edge of the bubblesort graph. Thus, $U_{j'} = U_j \circ (k-1,k)$ for some $2 \le k \le n-1$. If $i \ne k-1$ and $i \ne k$, then $\phi(N(i,j')) = \phi(N(i,j))\circ(k,k+1)$. If $i = k-1$, then $\phi(N(i,j')) = \phi(N(i,j))\circ(1,k+1)$. If $i = k$, then $\phi(N(i,j')) = \phi(N(i,j))\circ(1,k)$.

As a result, we can obtain that a $T(n\times(n-1)!)$ can be embedded into a $BS(n)$ with dilation 2 by combining the proofs of the CASE 1 and CASE 2.

Finally, the average dilation, $\frac{3}{2} - \frac{1}{n}$, can be easily derived from this proof directly [4]. □

In order to compare with the mesh embedding capacity of the star graph under dilation 1 and 2, we list the results as follows. First, there is no dilation 1 embedding of a k-dimensional mesh into an $S(n)$ for all $k > 1$ and for all n [9, 17]. Secondly, 2-dimensional mesh of size $2\times(n!/2)$ can be embedded into an $S(n)$ with dilation 2, which follows from the fact that 2-dimensional mesh of size $2\times(N/2)$ can be embedded into any N-node Hamiltonian graph [9]. However, this kind of mesh is rarely used because of its low bisection width (2) and long diameter $(n!/2)$. On the other hand, developing the low dilation embedding of a 2-dimensional mesh of the size $n\times(n-1)!$ into an $S(n)$ is useful and important since A.Menn and A.K.Somani [14] make use of this kind of mesh embedding to derive the efficient shearsort algorithm on the $S(n)$. Thus, from Theorem 3.2, we can know that the mesh embedding capacity of the bubblesort star graph is better than that of the star graph.

3.2 Cube Embeddings

In this subsection, we will propose a recursive method to embed the multiple copies of the hypercube into a bubblesort star graph with all faults recovery capacity as well as the constant expansion and dilation 1 and/or 2.

Let $u = (b_{k-1}b_{k-2}\dots b_1b_0)$ and $v = (d_{k-1}d_{k-2}\dots d_1d_0)$ be two arbitrary binary strings of length k, then the Hamming distance between u and v is denoted by $HD(u,v)$ and $HD(u,v) = \sum_{i=0}^{k-1}(b_i \oplus d_i)$, where $\oplus$ is the Exclusive-OR function. The hypercube with dimension k is denoted by $H(k)$ and its vertex set and edge set are defined as $V_{H(k)} = \{(b_{k-1}b_{k-2}\dots b_1b_0) : b_i \in \{0,1\}\}$ and $E_{H(k)} = \{(u,v) : u, v \in V_{H(k)},$ and $HD(u,v) = 1\}$, respectively.

Dilation One Embeddings

Lemma 3.3 [2, 10] $H(k) \sqsubseteq B(2k)$ for all k.

Lemma 3.4 If $H(k) \sqsubseteq BS(n)$ and $H(k') \sqsubseteq B(n')$, then $H(k+k') \sqsubseteq BS(n+n')$.

Proof. Let $BS(n) = \langle V_{BS(n)}, E_{BS(n)}\rangle$ and $V_{BS(n)} = Perm(\mathcal{S}_n)$. Let $B(n') = \langle V_{B(n')}, E_{B(n')}\rangle$ and $V_{B(n')} = Perm(\mathcal{S}_{n'})$, where $\mathcal{S}_n \cap \mathcal{S}_{n'} = \emptyset$. Let φ represent the embedding of the k-cube into the bubblesort star graph $BS(n)$ and ψ denote the embedding which maps the vertex set $V_{H(k')}$ to the set $V_{B(n')}$. Now, we define the embedding $\phi : V_{H(k+k')} \to V_{BS(k+k')}$ as $\phi(u) = \varphi(v)\|\psi(w)$, where $v \in V_{H(k)}$, $w \in V_{H(k')}$, and $u = v\|w$. Obviously, $u \in V_{H(k+k')}$ and $\phi(u) \in V_{BS(n+n')}$. The embedding ϕ is hence one-to-one, which follows from the fact that both φ and ψ are one-to-one.

Let $u',\ w' \in V_{H(k)}$, $u'',\ w'' \in V_{H(k')}$ and $u = u'\|u''$, $w = w'\|w''$. Thus $u,\ w \in V_{H(k+k')}$ and provided that $(u,w) \in E_{H(k+k')}$, the relation between u and w is exactly one of the following two cases:

CASE 1: $HD(u',w') = 1$ and $HD(u'',w'') = 0$.

Since $H(u',w') = 1$, by the definition of the embedding φ, we can know that $\varphi(u') = \varphi(w')\circ(i,j)$ for some i and j, where $i = 1$ and $2 \le j \le n$, or $i = j-1$ and $2 \le j \le n$. Besides, $\psi(u'') = \psi(w'')$. Thus, we can immediately obtain that $\phi(u) = \phi(w)\circ(i,j)$.

CASE 2: $HD(u',w') = 0$ and $HD(u'',w'') = 1$.

Since $H(u'',w'') = 1$, we have $\psi(u'') = \psi(w'')\circ(i-1,i)$ for some $2 \le i \le n'$. Additionally, $\varphi(u') = \varphi(w')$. From these expressions, it follows that $\phi(u) = \phi(w)\circ(n+i-1, n+i)$.

Consequently, the above embedding, ϕ, of the $H(k+k')$ into the $BS(n+n')$ has a *dilation one*. □

Theorem 3.5 $H(1) \sqsubseteq BS(2)$, $H(2) \sqsubseteq BS(3)$, $H(3) \sqsubseteq BS(4)$, and $H(k) \sqsubseteq BS(2k-2)$ for all $k \ge 3$.

Proof. For the first two parts of this theorem, the mappings can be obtained by inspection. As for $H(k) \sqsubseteq BS(2k-2)$, it can be proved by induction on k using Lemmas 3.3 and 3.4. □

Now, we consider the multiple network embeddings. Let G_1 and G_2 be two graphs and if we can embed two disjoint copies of the graph, G_1 and G_2, into an H with the same dilation d, then we write $(G_1 \uplus G_2) \sqsubseteq H^d$. We denote $G_1 \uplus G_2 = 2\cdot G$ if both G_1 and G_2 are isomorphic to G. According to the embedding method proposed in Lemma 3.4, we have the following lemma [4].

Lemma 3.6 If $H(k) \uplus H(k') \sqsubseteq BS(n)$, and $H(k'') \sqsubseteq B(n')$, then $H(k+k'') \uplus H(k'+k'') \sqsubseteq BS(n+n')$.

Let Π denote the permutation function which is one-to-one and maps the set $\mathcal{S}_n$ to $\mathcal{S}_n$. Let $\sigma, u \in Perm(\mathcal{S}_n)$ be two permutations, the operation $\Pi_\sigma(u)$ reorders the positions in u according to the permutation σ. That is, $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ and $u = u_1u_2\ldots u_n$, then $\Pi_\sigma(u) = \sigma_{u_1}\sigma_{u_2}\ldots\sigma_{u_n}$. Given the subgraph G of the star graph $S(n)$ or bubblesort star graph $BS(n)$, $\Pi_\sigma(G)$ denotes the graph whose vertices are reordered by applying the operation $\Pi_\sigma(V_G)$. Thus, we have the following lemma [4].

Lemma 3.7 Let G be the subgraph of the $S(n)$ or $BS(n)$, then $\Pi_\sigma(G)$ preserves the topology of the subgraph G for any permutation σ.

Lemma 3.8 If $k\cdot H(n) \sqsubseteq BS(2n-2)$, then $k\binom{2n}{2}\cdot H(n+1) \sqsubseteq BS(2n)$.

Proof. From Theorem 3.5, we know that $H(n) \sqsubseteq BS(2n-2)$. Let $V_{BS(2n-2)} = Perm(\mathcal{S}_{2n-2})$ and $V_{B(2)} = Perm(\mathcal{S}'_2)$, where $\mathcal{S}_{2n-2}\cap\mathcal{S}'_2 = \emptyset$. By Lemma 3.6 and the hypothesis of this lemma, we have $k\cdot H(n+1) \sqsubseteq BS(2n)$. However, according to Lemma 3.7 and the method proposed in Lemma 3.4, we have $\binom{2n}{2n-2} = \binom{2n}{2}$ possible choices for $2n-2$ symbols to construct the disjoint isomorphic embedded $(n+1)$-cubes in the $BS(2n)$. Thus, it follows that $k\binom{2n}{2}\cdot H(n+1) \sqsubseteq BS(2n)$. □

Theorem 3.9 For $n \geq 3$, we can embed $\frac{(2n-2)!}{3\times 2^{n-1}}$ disjoint copies of $H(n)$ into a bubblesort star graph $BS(2n-2)$ with a dilation equals to one and expansion equals to 3/2.

Proof : We prove this theorem by induction on n. For the basis: $n = 3$, we can obtain $2\cdot H(3) \sqsubseteq BS(4)$ by inspection. Assume that $\frac{(2n-4)!}{3\times 2^{n-2}}$ disjoint copies of $H(n-1)$ can be embedded into a $BS(2n-4)$ for $n \geq 3$. For the inductive step, by Lemma 3.8, it follows that we can embed $\frac{(2n-4)!}{3\times 2^{n-2}} \times \binom{2n-2}{2} = \frac{(2n-2)!}{3\times 2^{n-1}}$ disjoint copies of $H(n)$ into a $BS(2n-2)$. Lastly, the expansion of the embedding is

$$\frac{3\times 2^{n-1}}{(2n-2)!} \times \frac{(2n-2)!}{2^n} = \frac{3}{2}\,. \qquad \square$$

Now, we will propose a fault recovery scheme to achieve the fault tolerant embeddings. Every node in the host graph is given a state: active state, inactive state, spare state, or faulty state. The recovery cost depends on the number of states needed to be changed. Thus the simpler the recovery scheme is, the more efficient the recovery is. Besides, our fault tolerant embedding scheme is based on the partial fault model; that is, owing to the development of the technology on multicomputers, the router equipped in the partially faulty processor can also be used to pass the message or flits to other healthy processors.

Theorem 3.10 For $n \geq 3$, we can embed $\frac{(2n-2)!}{3\times 2^n}$ disjoint copies of $H(n)$ into a bubblesort star graph $BS(2n-2)$ with unit dilation such that each embedded node in the $BS(2n-2)$ has a distinct neighboring node as its own spare node. Thus, if faulty nodes in each copy of the embedded n-cube allow of no adjacency, then t faults can be recovered in t steps with dilation two. Above all, all faults in each copy of the embedded n-cube can be recovered in 2^n steps with dilation three.

Proof. We divide the proof of this theorem into two parts. *First*, we make and prove the CLAIM 1 that we can embed $\frac{(2n-2)!}{3\times 2^n}$ disjoint copies of $H(n)$ into a $BS(2n-2)$ with unit dilation. Besides, for all $u \in V_{H(n)}$, the embedded node $\phi(u)$ in $BS(2n-2)$ has its own spare node $\Phi(u)$ such that $dist(\phi(u), \Phi(u)) = 1$. *Secondly*, we will prove the CLAIM 2 that if faulty embedded nodes are not allowed to be adjacent, then t faults can be recovered in t steps with dilation 2; otherwise t faults can be recovered in t steps with dilation three.

First, we prove the CLAIM 1 by induction on n.
Induction base: For $n = 3$, the validity of this claim can be verified by the following table.

Table I, $\phi(V_{H(3)})$ and $\Phi(V_{H(3)})$

$u \in V_{H(3)}$	000	001	010	011
$\phi(u) \in V_{BS(4)}$	1234	3214	1324	2314
$\Phi(u) \in V_{BS(4)}$	2134	4213	3124	4312
$u \in V_{H(3)}$	100	101	110	111
$\phi(u) \in V_{BS(4)}$	4231	3241	4321	2341
$\Phi(u) \in V_{BS(4)}$	2431	1243	3421	1342

Induction hypothesis: Assume that we can embed $\frac{(2n-4)!}{3\times 2^{n-1}}$ disjoint copies of $H(n-1)$ into a $BS(2n-4)$ with dilation one. Besides, we have the embedding, φ, which maps the vertex set of $H(n-1)$ into the $V_{BS(2n-4)}$ such that for all $u \in V_{H(n-1)}$, the embedded node $\varphi(u)$ in the $BS(2n-4)$ has its own spare node $\Phi(u)$ and $dist(\varphi(u), \Phi(u)) = 1$.
Induction step: By Lemma 3.3, we have the embedding $\psi : V_{H(1)} \to V_{B(2)}$. According to the method proposed in Lemma 3.4, we define the embedding, ϕ, by mapping the node $v = u\|w$ in the $H(n)$ into the node $\phi(v) = \varphi(u)\|\psi(w)$ in the $BS(2n-2)$, where $u \in V_{H(n-1)}$ and $w \in V_{H(1)}$. Besides, let the spare node of $\phi(v)$ be $\Phi(u)\|\psi(w)$. Since the mappings φ and ψ are one-to-one, each embedded node in the $BS(2n-2)$ has its own spare node and their distance is $dist(\phi(v), \Phi(u)\|\psi(w)) = dist(\varphi(u)\|\psi(w), \Phi(u)\|\psi(w)) = dist(\varphi(u), \Phi(u)) = 1$. In addition, by Lemma 3.8, we can embed $\frac{(2n-4)!}{3\times 2^{n-1}} \times \binom{2n-2}{2} = \frac{(2n-2)!}{3\times 2^n}$ disjoint copies of $H(n)$ into a $BS(2n-2)$. This claim is thus proved.

Secondly, we submit the proof of the CLAIM 2.

Let $v_1 = u_1\|w_1$ and $v_2 = u_2\|w_2$ be any two neighboring nodes in the n-cube, where $u_1, u_2 \in V_{H(n-1)}$ and $w_1, w_2 \in V_{H(1)}$. Thus the embedded nodes of v_1 and v_2 in $V_{BS(2n-2)}$ are $\phi(v_1) = \varphi(u_1)\|\psi(w_1)$ and $\phi(v_2) = \varphi(u_2)\|\psi(w_2)$ separately, and their respective spare nodes are $\Phi(u_1)\|\psi(w_1)$ and $\Phi(u_2)\|\psi(w_2)$. Since the embedding, ϕ, of $H(n)$s into the $BS(2n-2)$ has a dilation one, the embedded nodes, $\phi(v_1)$ and $\phi(v_2)$, will be adjacent in the $BS(2n-2)$. Therefore, if only $\phi(v_1)$ becomes faulty and is replaced by its spare node, then $dist(\Phi(u_1)\|\psi(w_1), \phi(v_2)) = 2$. If both $\phi(v_1)$ and $\phi(v_2)$ become faulty and are replaced by their respective spare nodes, then $dist(\Phi(u_1)\|\psi(w_1), \Phi(u_2)\|\psi(w_2)) = 3$. Since every embedded node has its own spare node, this claim thus follows.

Combining the proofs of CLAIM 1 and 2, this theorem is thus reached. □

In order to compare with the cube embedding capacity of the star graph under unit dilation, we list the result as follows. M. Nigam et al. [16] indicated that there is no dilation 1 embedding of an $H(k)$ into an $S(n)$, where $k > 1$ and for any n. Clearly, the cube embedding capacity of the bubblesort star graph is much better than that of the star graph since we can embed multiple disjoint copies of the n-cube into the $BS(2n-2)$ with constant expansion as well as all faults recovery capacity.

Dilation Two Embeddings

Using the same method proposed in Lemma 3.4, we have the following lemmas.

Lemma 3.11 If $H(k) \sqsubseteq B^2(n)$ and $H(k') \sqsubseteq B^2(n')$, then $H(k+k') \sqsubseteq B^2(n+n')$.

Lemma 3.12 If $H(k) \sqsubseteq BS^2(n)$ and $H(k') \sqsubseteq B^2(n')$, then $H(k+k') \sqsubseteq BS^2(n+n')$.

Lemma 3.13 $H(1) \sqsubseteq B^2(2)$, $H(2) \sqsubseteq B^2(3)$, and $H(3) \sqsubseteq B^2(4)$. In addition, $H(3k-1) \sqsubseteq B^2(4k-1)$, $H(3k) \sqsubseteq B^2(4k)$, and $H(3k+1) \sqsubseteq B^2(4k+2)$ for all k.

Proof: For the first three parts of this theorem, the mappings can be obtained by inspection. The remainder of this lemma can be proved by induction on k using Lemma 3.11. □

Lemma 3.14 If $H(k) \sqsubseteq B^2(n)$, then $H(k + \lfloor \log_2(n+1) \rfloor) \sqsubseteq BS^2(n+1)$.

Proof: Let $\mathcal{S}_n = \{0, 1, 2, \ldots, n-1\}$, $B(n) = \langle V_{B(n)}, E_{B(n)} \rangle$, and $V_{B(n)} = Perm(\mathcal{S}_n)$. Let $BS(n+1) = \langle V_{BS(n+1)}, E_{BS(n+1)} \rangle$, and $V_{BS(n+1)} = Perm(\mathcal{S}_{n+1})$, where $\mathcal{S}_{n+1} = \{n\} \cup \mathcal{S}_n$. Let φ denote the embedding of an $H(k)$ into an $B(n)$ with dilation 2. Let $p = \lfloor \log_2(n+1) \rfloor$, $u \in V_{H(k+p)}$, and $u = v \| w$; where $v \in V_{H(k)}$ and $w \in V_{H(p)}$. Then we define the embedding, $\phi : V_{H(k+p)} \rightarrow V_{BS(n+1)}$, by mapping the node $u = v\|w$ in the $H(k+p)$ to the node $(n\|\varphi(v)) \circ (1, 1 + Int(w))$ in the $BS(n+1)$, where $Int(w)$ represent the integer value of the binary string w.

By Lemma 3.1, it follows that the embedding ϕ is one-to-one. Therefore, we will examine the dilation of the embedding ϕ.

Let u', $w' \in V_{H(k)}$, u'', $w'' \in V_{H(p)}$, and $u = u'\|u''$, $w = w'\|w''$. Thus if $u, w \in V_{H(k+p)}$ and u, w are neighbors in the $H(k+p)$, the relation between u and w is exactly one of the following two cases: (1)$HD(u', w') = 1$ and $HD(u'', w'') = 0$ or (2)$HD(u', w') = 0$ and $HD(u'', w'') = 1$

CASE 1: $HD(u', w') = 1$ and $HD(u'', w'') = 0$.

Thus, the distance between $\varphi(u')$ and $\varphi(w')$ is one or two since $HD(u', w') = 1$. Without loss of generality, we only consider the condition that $dist(\varphi(u'), \varphi(w')) = 2$ since the proof of this lemma under the condition that $dist(\varphi(u'), \varphi(w')) = 1$ can be verified by the same argument. Therefore, we let $\varphi(u') = \varphi(w') \circ (i-1, i) \circ (j-1, j)$ for some $2 \leq i, j \leq n-1$ and $i \neq j$. Let $A = 1 + Int(u'')$. We realize that $1 + Int(u'') = A = 1 + Int(w'')$ since $HD(u'', w'') = 0$. Assume that $i < j$, the relation among A, i, and j is necessarily one of the following subcases.

Subcase 1.1: $A \notin \{i, i+1, j, j+1\}$. Thus $\phi(u) = \phi(w) \circ (i, i+1) \circ (j, j+1)$.

Subcase 1.2: $A \in \{i, i+1, j, j+1\}$ and $i+1 \neq j$. Thus, if $A = i$, then $\phi(u) = \phi(w) \circ (1, i+1) \circ (j, j+1)$. If $A = i+1$, then $\phi(u) = \phi(w) \circ (1, i) \circ (j, j+1)$. Likewise, if $A = j$ or $j+1$, the distance between $\phi(u)$ and $\phi(w)$ in the $BS(n+1)$ is still equal to two.

Subcase 1.3: $A \in \{i, i+1, j, j+1\}$ and $i+1 = j$. Therefore, if $A = i$, then $\phi(u) = \phi(w) \circ (1, j) \circ (1, j+1)$. If $A = j$, then $\phi(u) = \phi(w) \circ (1, j+1) \circ (1, i)$. In addition, if $A = j+1$, the distance between $\phi(u)$ and $\phi(w)$ equals to two as well.

CASE 2: $HD(u', w') = 0$ and $HD(u'', w'') = 1$.

In this condition, $\varphi(u') = \varphi(w')$. Let $A_1 = 1 + Int(u'')$ and $A_2 = 1 + Int(w'')$. It can be verified that $A_1 \neq A_2$ since $HD(u'', w'') \neq 0$. Therefore, we can easily obtain that $\phi(u) = \phi(w) \circ (1, A_2) \circ (1, A_1)$.

Finally, we can obtain that an $H(k+p)$ can be embedded into a $BS(n+1)$ with dilation two by combining the proofs of the above two cases. □

Theorem 3.15 $H(3 \cdot 2^{n-2} + n - 1) \sqsubseteq BS^2(2^n)$ for all $n \geq 2$.

Proof: This formula can be derived from Lemmas 3.13 and 3.14. From Lemma 3.13, we can realize that $H(3k-1) \sqsubseteq B^2(4k-1)$. Let $4k - 1 = 2^n - 1$ and apply Lemma 3.14, this theorem thus follows. □

By the same arguments on Lemmas 3.6 and 3.8, we have the following lemmas.

Lemma 3.16 If $H(k) \uplus H(k') \sqsubseteq B^2(n)$ and $p = \lfloor \log_2(n+1) \rfloor$, then $H(k+p) \uplus H(k'+p) \sqsubseteq BS^2(n+1)$.

Lemma 3.17 If $k \cdot H(n) \sqsubseteq B^2(m)$ and $k' \cdot H(n') \sqsubseteq B^2(m')$, then $kk' \binom{m+m'}{m} \cdot H(n+n') \sqsubseteq B^2(m+m')$.

Like Theorem 3.9 and Lemma 3.13, the following lemma and theorems can be proved by induction on k using Lemmas 3.11, 3.13, 3.16, and 3.17.

Lemma 3.18 $\frac{(4k)!}{12^k} \cdot H(3k) \sqsubseteq B^2(4k)$ and $\frac{(4k-1)!}{6 \cdot 12^{k-1}} \cdot H(3k-1) \sqsubseteq B^2(4k-1)$.

Theorem 3.19 Let $k = 2^{n-2}$. For all $n \geq 2$, we can embed $\frac{(4k-1)!}{6 \times 12^{k-1}}$ disjoint copies of $H(3 \cdot 2^{n-2} + n - 1)$ into a bubblesort star graph $BS(2^n)$ with dilation two.

By the same argument on Theorem 3.10, we can achieve the fault tolerant cube embedding on the bubblesort star graph under dilation two. From Theorem 19, we know that multiple disjoint copies of the Δ-cube can be embedded into a $BS(2^n)$ with dilation 2, where $\Delta = 3 \cdot 2^{n-2} + n - 1$. Besides, the ratio of the dimension of the bubblesort star graph to that of the hypercube is $\frac{4}{3}$ when n is sufficiently large. In order to compare with the cube embedding capacity of the star graph under two dilation, we list the result as follows. M. Nigam et al. [16] indicated that an $H(n)$ can be embedded into a star graph $S(2^n)$ with dilation 2. Clearly, the expansion of this embedding is too intolerably large for us to accept. Thus, even under the condition of two dilation, the cube embedding capacity of the bubblesort star graph is still better than that of the star graph.

4 Conclusions

In this paper, we have presented and analyzed a new interconnection network called the bubblesort star graph, which is the merger of the bubblesort graph and star graph. We have found that not only the bubblesort star graph inherits the most superiority of the star graph but also it improves some performance of the star graph such as smaller average diameter, higher bisection width, lower traffic density, shorter broadcasting time, stronger network resilience and so forth [4]. Above all, we have known that a mesh can be embedded into a bubblesort star graph with dilation two and unit expansion; besides, multiple disjoint copies of the hypercube can also be embedded into a bubblesort star graph with all faults recovery capacity as well as constant expansion and dilation one or two. This reflects the fact that the embeddability of the bubblesort star graph is much better than that of the star graph. Consequently, we may expect that the bubblesort star graph is an attractive and versatile network for the parallel algorithms design.

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