Ch24 solution

24.3-4

Professor Gaedel has written a program that he claims implements Dijkstra's algorithm. The program produces $v.d$ and $v.\pi$ for each vertex $v \in V$. Give an O(V+E) time algorithm to check the output of the professor's program. It should determine whether the d and π attributes match those of some shortest-paths tree. You may assume that all edge weights are nonnegative.

1. Verify that $s.d = 0$ and $s.\pi = NIL$
2. Verify that $v.d = v.\pi.d + w(v.\pi, v)$ for all $v \neq s$
3. Verify that $v.d = \infty$ if and only if $v.\pi = NIL$ for all $v \neq s$
4. If any of above verification tests fail, declare the output to be incorrect. Otherwise, run one pass of Bellman-Ford, i.e. relax each edge(u,v) ∈ E one time. If any values of $v.d$ changes, then declare the output to be incorrect; otherwise, declare the output to be correct.

24.3-8

Let $G = (V, E)$ be a weighted, directed graph with nonnegative weight function $w: E \rightarrow \{0,1, \ldots, W\}$ for some nonnegative integer W . Modify Dijkstra's algorithm to compute the shortest paths from a given source vertex s in $O(WV + E)$ time.

Dijkstra's algorithm is given as follows:

Dijkstra(G, w, s)

1. INITIALIZE-SINGLE-SOURCE(G, s)
2. $S \leftarrow \{\}$
3. $Q \leftarrow V[G]$
4. while Q != {}
    a. do u $\leftarrow$ EXTRACT-MIN(Q)
    b. S $\leftarrow$ S∪{u}
    c. For each vertex v∈Adj[u]
        i. Do RELAX(u, v, w)

The running time of Dijkstra's algorithm depends on the implementation of the min-priority queue. In Dijkstra's algorithm, we process those vertices closest to the source vertex first. Because each edge has at most weight W, we know the maximum possible value of the longest path in the graph is (V-1)W. We can prioritize the vertices based on their d[] values. Remember, d[v] is the shortest path from the source to vertex v.

The queue consists of (V-1)W buckets. Vertex v can be found in bucket d[v]. Since all other than the source have d[v] value between 1 and (V-1)W, so they can be found in buckets 1,…,(V-1)W. If s is the source vertex, d[s] = 0. So, s can be found in bucket 0. INITIALIZE-SINGLE-SOURCE ensures that for all vertices v other than the root, d[v] is initialize to ∞. The final bucket holds all vertices whose d[]-values are infinity (all undiscovered vertices).

After initializing all of the vertices, we scan the buckets from 0 to (V – 1)W. When a non-empty bucket is encountered, the first vertex is removed, and all adjacent vertices are relaxed. This step is repeated until we have reached the end of the queue—in O(WV) time. Since we relax a total of E edges, the total running time for this algorithm is O(VW + E)

24.3-10

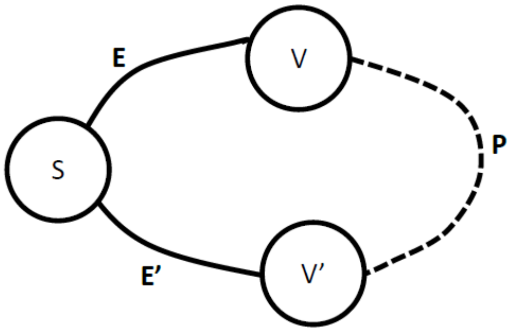
Suppose that we are given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, all other edge weights are nonnegative, and there are no negative-weight cycles. Argue that Dijkstra's algorithm correctly finds shortest paths from s in this graph.

According to the question, we know that there is no negative weight cycle in the graph G and all negative weight edges are connected to the source vertex s. Therefore, we just need to prove that below:

> If some vertex $v \neq s$ is connected with some negative weight edge e, the shortest path from s to v must cover the negative weight edge e.

Next, we could apply the Theorem 24.6 and know that Dijkstra's algorithm is still correct.

Prove: (By contradiction)



// Both E and E' are negative weight edges

Assume that the shortest path from S to V is $\mathrm{S} \xrightarrow{\mathrm{E}'} \mathrm{V}' \xrightarrow{\mathrm{P}} \mathrm{V}$ instead of $\mathrm{S} \xrightarrow{\mathrm{E}} \mathrm{V}$. Then we could get the equation $\mathrm{E}' + \mathrm{P} < \mathrm{E}$.

$\quad \mathrm{E}' + \mathrm{P} < \mathrm{E}$

$\Rightarrow \mathrm{E} + \mathrm{E}' + \mathrm{P} < 2\mathrm{E} < 0$ // negative cycle

Contradiction occurs.